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Short, but yet Plain

ELEMENTS

OF

GEOMETRY.

SHEWING

How by a Brief and Easy Method, most of what is Necessary and Useful in EUCLID, ARCHIMEDES, APOLLONIUS, and other Excellent *Geometricians*, both Ancient and Modern, may be Understood.

Written in *French*

By F. IGNAT. GASTON PARDIES.

And render'd into *English*,

By JOHN HARRIS, D. D.

And Secretary to the *Royal Society*.

The EIGHTH EDITION.

L O N D O N,

Printed for A. WARD, at the *King's-Arms* in *Little-Britain*. MDCCXLVI.

Short, but not Plain

ELEMENTS

OF

GEOMETRY

THE FOLLOWING

How by a Rule and Two Compasses, to draw a Circle, and to divide it into any Number of Parts, and to draw a Square, and a Rectangle, and a Rhombus, and a Rhomboid, and a Trapezium, and a Trapezoid, and a Polygon, and to find the Area of any of them.



By F. LONAT, GASTON PARBIE.

By JOHN HARRIS, D.D.

The Eighth Edition.

L O N D O N

Printed for A. W. at the Royal Academy, in the Strand.
R. G. W. 1804.



T O

My Worthy Friend,
CHARLES COX, *Esq;*
Member of Parliament for the
Burgh of *Southwark*.



AMONG the many Obligations
You have conferred on me, I ac-
count it not the least, that you
have given me a *Rise* to revive
my *Mathematical Studies*; in
which, as I have formerly spent some Time,
so I know of no more *useful Way* of employ-
ing my leisure Hours.

And indeed, Sir, the *Diversion* and *Ad-
vantage* I have lately reaped from them, hath
(by the Divine Blessing) both supported me
under, and in a good Measure carried me
through such Pressures and Difficulties, as I
once almost despaired of surmounting.

The Epistle Dedicatory.

The *Mathematick Lecture* which You at first set up *gratis* in your *Burgh*, and which out of an uncommon Generosity, You did afterwards remove into the City of *London*, is a demonstrative Proof both of your sincere Endeavour to promote the Good of your Country, and also of your Capacity to do it the best Way. And as I have already, in a good Degree, so I hope to see such Effects from so noble a Design, as will render your Name justly honourable to Posterity, as well as this present Age. Sir, you know *your Self* and *Me* too well to take this for Flattery. 'Tis what *Truth*, *Justice*, and *Gratitude* oblige me to say.

I shall only add, that I am again glad of this Opportunity to shew the just Esteem I have of your Merit, and the equal Regard I have of your Friendship. *I am,*

S I R,

Your most obliged

14 NO 63

Humble Servant,

John Harris.



THE
TRANSLATOR
TO THE
READER.



After frequent Perusal, and mature Deliberation on this Book; I judge it to be the plainest, shortest, and yet easiest Geometry I have ever seen publish'd: And therefore I thought it very well worth my while to let it appear an eighth Time in our own Language, as it had already done twice before in the Latin Tongue. And 'tis so well esteem'd of by very competent Judges amongst Us, as to be read in our Universities, by Tutors to their Pupils: And also, which is not usual with Books of this Kind, the three first Impressions were sold off in little more than as many Years Time.

*As to the Translation; I have by no means obliged myself servilely to follow the French way of Expression; for indeed a literal Version of a Book out of any Language will be scarce intelligible in English. I have therefore all along aimed rather to give you F. Pardies's Sense, than his Words; and have made him speak what I judge he would have done, had he wrote in our Language. I have made no Scruple to add any thing that I saw necessary, to render him clear
and*

The TRANSLATOR &c.,

and intelligible; and particularly what follows, which was not in some of the former Editions, viz.

The second Book of Euclid, about the Power of Lines: The Mensuration of the Surfaces of Solids: Archimedes's Proportion of the inscribed Cone and Sphere to the circumscribing Cylinder: The Figure of the five regular Bodies: Several Additions and Improvements in the Doctrine of Proportion: The Mensuration of the Frustums of Pyramids and Cones: Some new Properties of a right angled Triangle, and of the Circle, &c. I have also left out some more of Pardies's Propositions, which, on repeated Experience in Teaching, I have found less useful; as also the Elements of Plain Trigonometry, which I had before added to his Ninth Book; because I have publish'd a small Treatise on that Subject by itself. And my chief Aim now hath been to lead the Learner into a little more abstracted and concise, tho' a most useful and universal Method of Demonstration; introducing now and then a little Algebra, that I may thereby ingage the Reader in a Love of, and Value for that most noble and wonderful Science; and to give him a good Foundation to build upon, and a sufficient Rise thereby to carry him into Fluxions, and the two Methods of Investigation and Demonstration, where he will find sufficient Satisfaction. Nor need he be discouraged at the Attempt, for 'tis well known, that I have taught several Persons to understand the Elementary Parts of all Mathematicks so well, that they have been able to go on every where, without the Assistance of any Master, in less than a Year's Time. 14 NO 63



PARDIES's *Advice to those who would Understand Geometry.*

I.  HE Learner should accustom himself to consider well the *Figures*, at the same Time that he reads the *Propositions*. There will be some Labour and Difficulty at first, but he will break thro' it in two or three Days.

II. He ought not to be discouraged, if he meets with some Things which he does not understand at first; Geometry is not so easy to be attained, as History.

III. If, after he has read and considered attentively any Proposition, he finds he don't understand it, let it be passed over; it will probably be intelligible by reading farther, or at least when he has gone over the whole, and has begun to read it over a-new. There are indeed many Things in Geometry, that will never be well understood at first reading over.

IV. The

Advice to those, &c.

IV. The Numbers which are within the Parenthesis, *v. gr.* (3. 14.) shew that the Matter there spoken of hath been proved elsewhere, *viz.* in this Instance, in the 14th *Article* of the third *Book*: And he should always consult the Article referred to, which will enable him to understand the Demonstration of what he reads.

V. When he meets with any Words which he don't understand, he may consult the Table at the End of the Book.

VI. 'Tis proper to have a Master at first, who may explain the Nature and Manner of the Demonstrations; by which Means any one will learn much easier, and sooner, than he can by reading by himself.

14 N 033



E L E.



ELEMENTS

OF

GEOMETRY.

BOOK I.

Of Lines and Angles.

1.  Y the Word *Quantity*, which in the General is the Subject of *Geometry*, we mean that whereby one Thing, being compared with another of the same Nature, may be said to be Greater or Less, Equal or Unequal: As Extension, *i. e.* Length, Breadth or Thickness, Number, Weight, Time, Motion, and all those Things which are capable of being so compared as to More or Less, are the Object of Geometry.

2. We design nevertheless to consider now only Extension; as being that which serves for an *Example* and *Rule* to measure all other Quantities by.

B

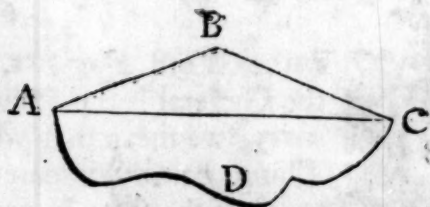
3. That

3. That Quantity which being supposed without any Breadth or Thickness, is extended only in Length, is called a *Line*. That which hath both Length and Breadth (but is supposed to have no Thickness) is called a *Surface* or *Superficies*: And that which hath Length, Breadth, and Thickness, is called a *Body* or *Solid*.

4. A *Point* is that, which is consider'd as having no manner of Dimensions; and as being indivisible in every respect. The Ends or Extremities of Lines, as also the Middle of them, are Points.

5. There are *Straight Lines*, and there are *Crooked* or *Curved* ones: Also there are Even and Plain Surfaces which are called *Planes*; and there are *Crooked* or *Curved* ones, which like a Vault (*or the Tilt of a Boat or Waggon*) are *Convex* above, and *Concave* below.

The Generation of Lines may easily be conceived to be made by the Motion or Fluxion of a Point, as A:



Which if it move directly from the Term A to the Term C, or go the nearest or shortest Way possible, it then forms what Geometers call a *Right* or *Straight Line*.

If it go first directly to B, and then also the nearest Way to C; it forms two Right Lines, AB and BC, which, taken together, are longer than the Line AC; and consequently, two Sides of any Triangle must be longer than the Third.

If

If the Point A move not in one or more right or strait Lines towards C, it must go crooked, and so will form a Curve or crooked Line, as ADC.

And from hence also 'tis plain, that any two Points, moving with equal Velocity, will in the same Time generate equal Lines.

6. When two Lines meet in a Point, the *Aper-
ture*, *Distance*, or *Inclination* between them, is call'd an *Angle*. Which, when the Lines forming it are right or strait ones, is called a *Rectilineal Angle*; as A. But if they are crooked, 'tis called a *Curvilinear One*; as B. And when one is strait and the other crooked, 'tis called a *Mix'd Angle*; as C.

N. B. *The Lines, forming any Angle, are called its Legs.*



7. That Angle is said to be less than another, whose Legs are more inclined to (or nearer to) each other. Let there be two Lines AB and AC meeting in the Point A. If you imagine those Lines to be moveable like the Legs of a Pair of Compasses, and yet fastened together in A, as with a Joint, 'tis easy then to conceive, that the further they are opened, or parted from one another, the greater will be the Angle between them: As on the contrary, the nearer they are brought together, the more they will incline towards each other, and so the Angle between them must be the less.



8. It must therefore be observed, that the Quantity of Angles is by no means measured by the Length of their Legs, but by their Inclination. Thus, *v. gr.* the Angle B is bigger than A; tho' the Legs of the Angle B, are much shorter than those

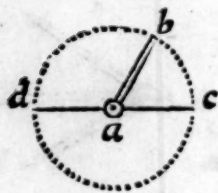


of A: But then those of A are much more inclined to each other, than those of B. And to apprehend this the better, imagine the Angle B to be put upon A, as you may conceive by the prick'd Lines about A,

which represent the Legs of B lying on it. For 'tis plain the Angle A will be easily contained within B; and that its Legs are much more inclined to one another, than those of B, and therefore it is less than B.

9. An Angle is usually marked by three Letters, of which the middlemost, and which always is placed at the Angular Point where the Lines meet, denotes the Angle. As in the Figure following *bac* denotes the Angle made by the two Lines *ba* and *ca* meeting in the Point *a*.

10. If we imagine the Line *ab* fastened by its End *a*, in the middle of the Line *dc*, but yet so as to be moveable to *a*, as on a Center:



If then you conceive it to be moved quite round, 'till it arrive at the Place where it began, the Point *b* will describe a Curve Line, which is usually called a Circle; but 'tis rather the *Cir-*

cumference of a Circle; for properly speaking, the Circle is the *Space* contained within the Circumference.

11. Any Part of the Circumference is called an Ark, as *bc*.

12. The Line *dc* (passing through the Center) and terminated by the Circumference, is called the *Diameter*,

meter, and divides the Circle into two equal Parts. Also every Right Line passing thro' the Center *a* (and terminated at each End by the Circumference) divides the Circle into two equal Parts, and will be a Diameter.

13. The Line *ab* or *ac*, or any other drawn from the Center to the Circumference, is called the Radius, or Semi-diameter.

14. All Radius's or Semi-diameters (of the same or equal Circles) are equal (As is plain from the Genesis of a Circle given in Art. 10.)

15. When the End *b* of the Radius *ab* is equally distant from the two Ends of the Diameter *dc*; That is, when the Point *b* is in the very middle of the Semi-circumference *dbc*; then will *ba* make two Angles with *dc* that are called *Right ones*, which are equal to one another, that is, the Angle *dab* is equal to *bac*. And if the Line *ba* be produced below to *e*, it shall then (with *dc*) make four Right Angles; and it will be another Diameter; which with the former *dc* will divide the Circle into four equal Parts.



16. Then those Lines are said to be perpendicular one to another, viz. *ba* to *dc*, and *da* to *be*.

17. But if *b* be nearer to one End of the Diameter (or Right Line) *dc*, than it is to the other, it is then said to fall on the other obliquely; and it makes with *dc* two Angles that are unequal: Of which the Lesser *bac* is called Acute, and the Greater *dab* is called Obtuse.



If the Line ab be produced to e , it will be a new Diameter, and will make below two other Angles:



So that in the whole here will be four Angles; of which those two that touch only in the Angular Point, as bac and ead ; as also, dab and eac , are called *Vertical*, or *Opposite* Angles. But those that have one Leg common to both, as dab and bac ; and bac and eac are called *Adjoining* or *Contiguous* Angles.

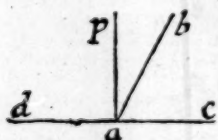
18. Those Angles, which (at equal Distances from the Angular Point) are subtended by equal Arks, are also equal themselves. As if the Ark bcb be proved equal to the Ark dea , then will the Angle bac be equal to dae .

19. The two *Contiguous* Angles, taken together, are always equal to two Right ones.

For as the Line dc is a Diameter, and therefore cuts the Circle into two equal Parts, the two Arks, db and bc , taken together, will be equal to a Semi-circle. Wherefore the two Angles, dab and bac , together will be equal to two Right ones, because they compleat the whole Semi-circle, as two Right ones do. (*Art. 15.*)

20. So that this Proposition is of universal Truth, That one Right Line, falling on another, makes the *Contiguous* Angles (together) equal to two Right ones.

For if the Lines are *Perpendicular* to each other, as



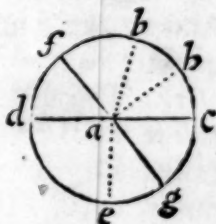
pa is to dc ; then 'tis plain the Angles must be Right (by the 15.) And if the Line fall obliquely, as ba doth, then indeed the Angles are unequal: But as much as the Obtuse one dab exceeds one Right

Angle, by so much is the Acute one bac exceeded by

by the other Right one. So that the Smallness of one is compensated by the Greatness of the other.

21. Hence also it follows conversely (for where-ever the Property is found, there the Thing is, in Geometry) that if two Angles, which have one Leg common to both, do make Angles equal to two Right ones, their other Legs do make but one Right Line. Let the Angles dab and bac be (together) equal to two Right ones. Then I say, that the Lines da and ac do join so together, as to make one Right Line (*vid. Fig. in Art. 17.*) which is clear from what hath been said. For if on the Center a you describe a Circle $dbce$, the two Arks db and bc will be equal to a Semi-circle, because the two Angles dab and bac are supposed to be equal to two Right ones. Wherefore the Lines da and ca will make a Diameter, and consequently be joined into one Right Line.

22. If from the Point a you draw several Lines, as ad , af , ab , ah , ag , &c. they will make diverse Angles; and all those Angles taken together, be they more or less, will be equal to four Right ones. For 'tis clear, all these Angles together do compleat the Circle $dbce$, whose Circumference they divide into as many Arks as they are Angles. Now all these together are equal to four Quarters of a Circle; which is as much as to say, that all the Angles are equal in the whole to four Right ones; for so many Right Angles do also compleat the Circle.



AXIOM I.

If to, or from equal Things, you add or subtract Equals, the Sums or Remainders will be equal.

23. The Vertical or Opposite Angles are equal. Let there be two Right Lines dac and bac (crossing or cutting one another in the Point a) I say, the Angle dae is equal to bac . For the Ark bd , with the



Ark bc , makes a Semi-circle; and so doth the same Ark bd with the Ark de . Therefore the Ark bc must be equal to de ; because the Ark db continues the same, whether it help to compleat the Semi-circle with de , or bc (wherefore being taken away from both, it

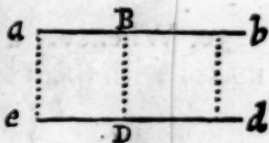
must leave the Ark de equal to bc . But if the Arks be equal, the Angles subtended by them must be so too, and therefore the Angle dae is equal to bac) And by the same Reason the Angle dab will be equal to eat .

24. The Circumference of every Circle is (suppose it to be) divided into 360 equal Parts, which are called Degrees: And every Degree into 60 Minutes, every Minute into 60 Seconds, every Second into 60 Thirds, and so on infinitely. And to determine the Quantity of every Angle, we compute the Degrees that the Ark, (which is its Measure) doth contain; *v. gr.* When we speak of an Angle of 90 Deg. we mean a Right Angle; because the Right Angle contains the fourth Part of the whole Circumference, which is 90 Deg. the fourth Part of 360. So an Angle of 60 Deg. is an Angle that contains two Thirds of a Right one.

25. Degrees

25. Degrees are marked either with Degr. or usually with a small Cypher over the last Figure, as 60° ; Minutes with a small Line, as $50'$; Seconds with two such, as $20''$; Thirds with three such, as $25'''$, &c. So that $25^{\circ} 32' 43''$, is to be read, 25 Degrees, 32 Minutes, 43 Seconds.

26. Two Lines are said to be parallel, when they run always equi-distant from each other. Thus the two Lines ab and ed are parallel, if they are equally distant from each other in ae , in BD , in ed , and in all other Places.

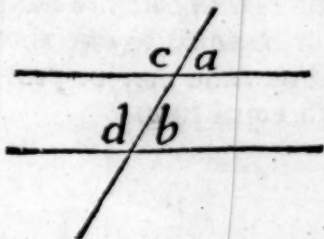


27. This Distance is always measured by a Perpendicular; as if from the Point a you imagine the Line ae to fall perpendicular on ec ; as also doth the Line bd on the same Line; we naturally conceive, that if those two Perpendiculars are of the same Length, or equal; the two Lines ab and ed are equally distant from each other in those two Places, which is self-evident, and needs no Proof.

28. Two parallel Lines being continued infinitely, yet can never meet: For being always equally distant, there may any where be drawn between them a Perpendicular equal to ae or bd , and consequently they can never meet.

29. Two parallel Lines have the same Inclination, one as the other, to any right Line that crosses them both.

That is the Angle a will always be equal to b , and c to d ; for the intersecting Line being supposed inflexible, as is the Case of all Ma-

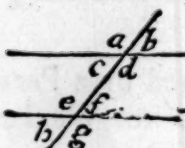


themathick Lines, it cannot bend to, or from one Parallel

Parallel more than it doth to or from the other : And neither of these Lines can alter its Position in respect of the Crossing Lines, for then the Parallelism would be destroyed, which contradicts the Supposition.

And this is the first Property of Parallel Lines.

30. Whenever a Right Line cuts two Parallels, it makes with them eight Angles: Of which four a, b, c, d are *external*; and the other four, e, f, g, h are *internal*. The Angles c and f , as also d and e , are called *Alternate*. The Angles e and a , as also f and b , are called the *internal, and opposite on the same Side*. And the Angles d, f , as also c and e , are called the *internal Angles on the same Side*.



AXIOM II.

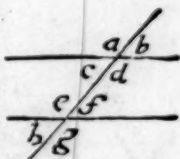
Things equal to a Third, are equal to one another.

31. The Alternate Angles c and f must be equal; and also e and d ; for c is equal to the Vertical Angle b , and b is equal to the internal one f , by the last *Prop.* Wherefore c and f , being both equal to b , must be equal to one another.

The same may be proved of c and d , which are both equal to a .

32. When a Line falls on two parallel ones, it makes the internal Angles on the same Side equal to two Right ones.

I say, the Angle d with f , is equal to two Right ones: Because f is equal to c (by 31.) and c and d together are equal to two Right ones (by 20.) Therefore f and d together must be equal to two Right ones, which was to be proved.



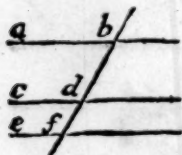
(The same Way may c and e together be proved equal to two Right ones; for c and d taken together are so (by 20) but d is equal to e (31) Therefore c and e are equal to two Right ones.)

33. One Proposition is called the *Converse* of another, when, after a Conclusion is drawn from something supposed in the *Converse* Proposition, that Conclusion is supposed; and then that which was in the other supposed, is now drawn as a Conclusion from it. For Example: We say here, if two Lines are parallel (*and another crosses them*) the Angles d and f together, are equal to two Right ones: Where we suppose the Lines to be parallel, and from thence conclude those Angles must be equal to two Right ones. But the *Converse* is thus: If the *internal Angles on the same Side*, d and f together, are equal to two Right ones; then those Lines are parallel: Where, after we have supposed the Angles equal to two right ones, we conclude the Lines are parallel.

34. *Converse* Propositions in this Case are very true; as that, if a Line cut two other Lines, and makes the alternate Angles equal, those two Lines are parallel; which I desire the Reader to remember.

35. If two Lines are parallel to a third Line, they are so to one another.

Let the Line ab be parallel to cd ; and let ef also be parallel to the same Line cd ; I say, ab is parallel to ef : For if you draw a



Line, as bdf cutting them all three, the Angle b will be equal to d (by 31.) and the same d will be equal to f (by 31.) because ef is also parallel to cd . Wherefore the Angle b must be equal to f : Because by Axiom 2. *If two Things are equal to a Third, they are so one to another.* But if the Angle $b = f$, then the Line ab is parallel to ef (by 34.)



14 N063



ELEMENTS OF GEOMETRY.



BOOK II.

Of Triangles.



1. *Figure* is a Space compassed round on all Sides. And if the Lines, which terminate it, are all Right ones, 'tis called a Rectilineal (or Right Lined) Figure: If they are crooked, 'tis called a Curvilineal; and if they are partly Right Lines, and partly Crooked, 'tis called a Mix'd Figure.

2. There are Plane Figures, which are Plane Surfaces, and there are Solid ones, which have three Dimensions. But we speak here only of Plane Surfaces, or Plane Figures.

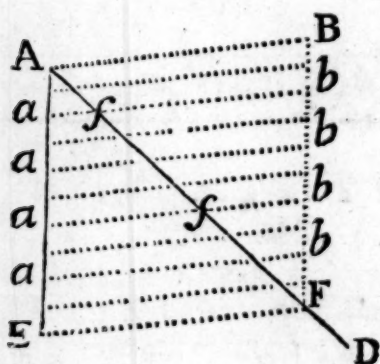
3. All

3. All the Lines which encompass any Figure, taken together, make that which is called the Circumference, Perimeter, or the Compass of the Figure.

4. Of all Curvilineal or Mix'd Plane Figures, in Common Geometry, we consider properly only the Circle, or a Part of a Circle, terminated on one Side by an Ark, and on the other by one or more Right Lines.

5. Of Rectilineal Figures, the most simple are Triangles, which are terminated by three Right Lines (*and no more*) making as many Angles.

If a Right Line (A B) having one of its Ends or Points (as A) in the Vertex or Top of the Angle E A D, be moved downwards, with a Motion always parallel to itself, so that the Point A shall always, keep in, or touch the Line A E, until it come to be



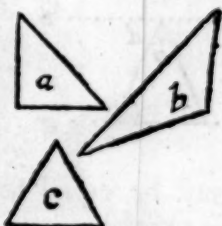
all of it within the Legs of the Angle EAD; that is, till it come to be in the Situation E F; that Line shall in its motion continually cut the Line A D, and at length describe the Triangle E A F within the Legs of the Angle; as also another equal to it

(A F B) on the other Side of the Line A D. The Parts of which latter Triangle shall continually decrease, as those of the former A E F, do continually increase. And the Line A B shall also describe with its whole Length the Quadrilateral Figure A E F B; which will be divided into two equal Parts by the Diagonal Line A F.

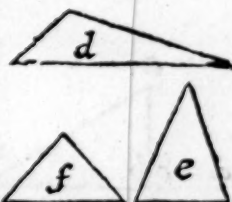
N. B.

N. B. The Line A B may be called the *Describent* and A E the *Dirigent*, because the latter directs the Motion of the former.

6. A Triangle, as *a*, which hath one Right Angle, is a *Right-angled* Triangle; if it have one Angle Obtuse, 'tis called an *Obtuse-angled* one, as *b*; and if all its three Angles are Acute, 'tis called an *Acute-angled* Triangle, as *c*.



7. If a Triangle have all its three Sides unequal, 'tis called a *Scalene*, as *d*. If it hath two Sides equal, 'tis called an *Isofceles*, as *e*; And if all the three Sides are equal, 'tis called an *Equilateral* one, as *f*.



8. When two Sides of a Triangle are considered, they may be called its *Legs*, and the third Side may then be called the *Base*. But any one Side may be called the *Base*, tho' we usually and most properly call that so, which lies parallel to the Horizon, and which is next to us.

9. In every Triangle, the three Angles, taken together, are equal to two Right ones.

Let the Triangle be *abc*: I say that the Angle *a* added to the Angle *c*, added to the Angle *b* (or the Sum of all three) are equal to two Right ones. For let *de* be drawn parallel to the *Base ac*, then will those two parallel Lines be cut by the Line *bc*; and consequently the alternate Angles *c* and *d* will be equal to each other (by 1. 31.) Moreover the Line *ba* falling on, or cutting the same Parallels *dc* and *ac*, will make the two internal Angles on the

the same Side equal to two Right ones; that is, a added to abe are equal to two Right Angles (by 1.

32.) But the Angle abe contains the two Angle b and d . So that

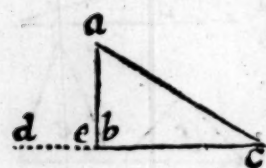
the Angle a added to b added to d , will be equal to two Right ones.

But c being equal to d , it will follow, that a added to b added to c ,

or the Sum of all three together, must be equal to two Right ones: Which was to be proved.

10. If any Side of a Triangle be produced, or drawn out, the external Angle

will be equal to the two internal opposite Angles (*taken together.*) Let the Triangle be



abc , whose Base cb draw out to d , by which means a new Angle, as e , will be made,

which is called the *External Angle* of that Triangle. Then I say, That that external Angle e , is equal to both the internal and opposite ones a and c .

For those Angles a and c , together with b , are equal to two Right ones (by the Precedent) and so also are e and b , by (1. 20.) wherefore e must be equal to a added to c , because together with b it makes two Right Angles, as they do. Q.E.D.

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COROLLARIES.

1. The Sum of the three Angles of all Triangles is the same.

2. No Triangle can have above one Right, or Obtuse Angle.

3. If

3. If in any Triangle, one Angle be Right, the other two must be Acute.

4. If in any Triangle there be one Angle equal to both the others, that must be a Right One.

5. If you know the Degrees of one Angle in any Triangle, you know the Sum of the other two, for 'tis what is wanting of 180° ; and if the Sum of any two be known, the Quantity of the Remainder is known.

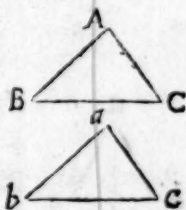
6. Hence if two Triangles have any two Angles respectively equal to one another, the remaining Angles must also be equal.

7. The Angle of an Equilateral Triangle is $\frac{2}{3}$ of two Right Angles, or $\frac{2}{3}$ of one Right Angle, equal to 60° .

8. Hence 'tis very easy to Trisect a Right Angle, by making on one of the Legs an Equilateral Triangle.

11. If a Triangle ABC hath two Sides, AB and AC, equal to two other ab and ac in another Triangle, and if also the Angle A be equal to a ; I say, the third Side BC shall be equal to bc ; the Angle B equal to b , the Angle C to c , and the whole Triangle ABC to abc .

For if we imagine the Triangle abc to be placed upon ABC, so that the Side ab shall lie exactly on its Equal AB: Then must the Side ac fall on its Equal AC, because the Angle a is equal to A, and so the Point c will fall on C, and b upon B, and the



C

whole

whole Triangle abc on the Triangle ABC ; because all things so exactly answer, that nothing of the upper Triangle can fall besides the under one.

12. Figures which do thus meet, fit, or answer to each other exactly, when they are placed one upon the other, are called Congruous Figures, *Quia mutuo sibi Congruunt*.

And therefore the third Axiom is, *Quæ sibi mutuo Congruunt sunt Æqualia*; i. e. Those Figures, which placed one upon another, do answer to, and cover one another exactly, are equal.

13. It is also true, That if a Triangle hath all its three Sides equal to the three Sides of another Triangle, all the Angles also in one, shall be equal to those in the other: And all the Space which one Triangle contains, shall be equal to that contained in the other: As if AB be equal to ab , AC to ac , and BC to bc : I say, that the Angle A shall be equal to a , B to b , and C to c ; and the whole Triangle ABC , to abc ; this needs no other Proof.

14. If the Angle A be equal to a , the Angle B to b , and the AB to ab : Then shall the Side AC be always equal to ac , BC to bc ; and the whole Triangle ABC to abc : Which is easy to prove by the precedent Propositions.

15. In every *Isoceles* Triangle, the Angles at the Base, opposite to the equal Legs, are equal.

Let the Triangle be abc , whose Legs ab and ac are equal: I say, the Angle b is also equal to c . For imagine the Base bc divided into two equal Parts in the Point d , then will the Line ad (which let be drawn) make of the whole two Triangles, abd and dac , which will have all three Sides in one, equal to those in the other: For



ab is equal to ac by the Supposition, and bd is equal to dc , and ad is common to both. Wherefore (by 2. 13.) the whole Triangle bad is equal to dac , and the Angle b is equal to c ; which was to be proved.

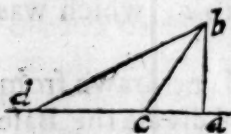
16. In an *Isoceles* Triangle, if a Line drawn from the Angle at the Top do (*bisect or*) divide the Base into two equal Parts, it is both perpendicular to the Base, and also bisects the Angle at the Top. For (*vid. Fig. precedent*) the Angle adc is equal to the Angle adb (by the last) and consequently they must be both Right ones; and therefore the Line ad is perpendicular to the Base bc (1. 15.) and the Angle dac will be equal to dab (by the last *Prop.*)

17. In every Triangle the Greater Side is always opposite to, or subtends the Greater Angle.

In the Triangle abc , let the Side bc be longer than ba ; then I say, the Angle bac subtended by the Greater Side bc , is bigger than the Angle c , which is subtended by the Lesser Side. For let bd be taken equal to ba , then will abd be an *Isoceles* Triangle; whose Angle bad will be equal to bda (2. 15.) But the Angle cab is bigger than bad (*The Whole being greater than the Part*) and therefore must be bigger than bda (*which is equal to bad*.) Now the Angle adb is an External Angle in respect of the little Triangle adc ; and therefore must be bigger than the Internal one c (by 2. 10. Wherefore the Angle bac being bigger than d , must certainly be bigger than c ; which was to be proved.



18. Of all Lines that can be drawn from a Point given to a Line given, the shortest is the Perpendicular; and they are all longer, according as they are farther distant from it. Let the given Line be ad , and the Point given b ; let ba be perpendicular to da ; let also bc and bd be drawn. I say, that ba is the shortest Line that can possibly be drawn from b ; and (for instance) is shorter than bc (or any other that can be assigned :) And I say also, that bd is longer than bc .



For in the Triangle bac , the Angle a is a Right One, and consequently bigger than either of the other; because they must necessarily be both Acute (by *Cor. 3. of Art. 10.*) Therefore the Side bc is longer than ba (2. 17.) as subtending a greater Angle.

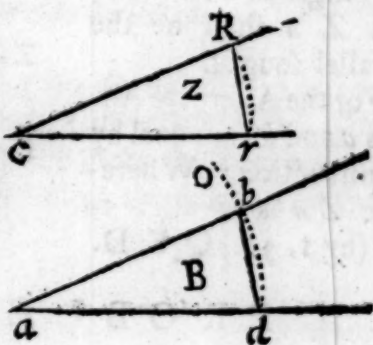
So also in the Triangle dbc , the Angle dcb is Obtuse, because the Angle bca is Acute: And consequently the Side db must be longer than cb , as subtending a greater Angle (2. 17.)

19. In every Triangle any two Sides taken together are longer than the third; because a Right Line is the nearest Distance between any two Points.

P R O B L E M I.

On a Line given $a d$, to make an Angle B , equal to a given one Z .

Place the Compasses in c , the Vertex of the given Angle, and describe the Ark $R r$; then keeping them at the same distance, set one Foot in a , one end of the given Line, and with the other describe the Ark $o b d$; set $R r$ from d to b ; and draw $a b$, so shall the Angle $b a d$ or B be equal to Z .



For the Legs of each are Radii of equal Circles, and the Line $b d$ was taken equal to $R r$; wherefore the whole Triangles $c R r$ and $a b d$ must be equal (by 13.) and consequently the Angle a equal to c .

P R O B L E M II.

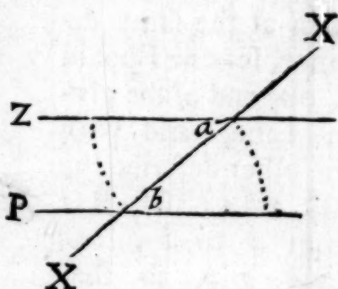
Hence the Practice of making all Sorts of Triangles, Equilateral, Isoscelar, or without any given Angles or Sides, will easily appear.

PROBLEM III.

A Right Line, as P, being given, to draw through a, a Point given, the Line Z a, parallel to it.

Through *a* draw any Line, as XX, making any Angle, as *b*, with the given Line; then make the Angle $Z a X = b$, and *Z a* shall be the Parallel sought.

For the Alternate Angles *a* and *b* are equal by Construction: Wherefore *Z a* is parallel to *P b* (by 1. 31.) Q. E. D.



PROBLEM IV.

To Bisect or Divide a given Line c b into two equal Parts in the Point a.

Open the Compasses to more than $\frac{1}{2}$ the Length of *b a c*, and with that distance make at each End of *b a c*, two Pairs of intersecting Arks, as at *e* and *d*: Then drawing the Line *ed*, it will bisect the given Line in *a*.



For the Triangles *bed* and *dec* are equal (by 2. 13.) Wherefore the Angle $abd = adc$. Therefore the Triangles *abd* and *adc* will be equal also (by 2. 11.) and consequently *ab* is = to *ac*. Q. E. D.

P R O-

PROBLEM V.

By much the same Method may a Perpendicular, as a d, be raised in the middle of any given Line, or one may be let fall from the Point e or d, to the given Line a b c, and the Demonstration is the same in all.

And after the same way of Practice may the given Angle b d c be bisected.

If setting one Foot of the Compasses in *d*, you take *db* equal to *dc*: And then setting the Compasses in *b* and *c*, strike the Arks intersecting each other in *e*: So shall *de* bisect the Angle required.





ELEMENTS

OF

GEOMETRY.

BOOK III.

Of Quadrilateral Figures and Polygons.

1.  THOSE Figures, whose Sides are four Right Lines, and those making four Angles, are called *Quadrilateral*, or four-sided Figures.

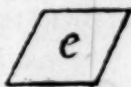
2. When the opposite Sides are parallel, the Quadrilateral Figure is called a *Parallelogram*, as *a*; but if not, 'tis called a *Trapezium*, as *B*.

3. When the Parallelogram hath all its four Angles Right, 'tis called a *Rectangled Parallelogram*; or, for brevity's sake, a *Rectangle*, as *c*: And if the Angles are right, and the Sides are all equal, 'tis called a *Square*, as *d*.

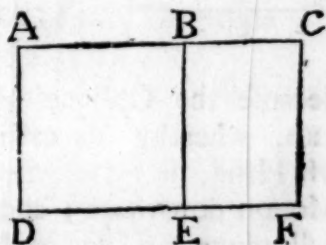
4. If

4. If a Parallelogram hath all its Sides equal, but its Angles unequal, then 'tis called a *Rhombus*, as *e*.

5. If a Parallelogram hath neither its Angles nor Sides all equal, 'tis called a *Rhomboides*, as *a*.



The Generation of all Parallelogrammick Figures will be easily conceived, if you suppose the Describent *A C*, to be carried or moved along the Dirigent *A D*, in a Position always parallel to itself in its first Situation. For then, if the Angle *A* which the Describent makes with the Dirigent, be a Right one, and *A B* be equal to *A D*, the Figure produced will be a Square. If *A C* be longer or shorter than *A D*, the Figure will be an Oblong or a Rectangle.



If the Angle at *A* be oblique, only a Parallelogram at large will be described: Which when the Describent is equal to the Dirigent, the Figure will be a *Rhombus*; if unequal to it, a *Rhomboides*.

C O R O L L A R I E S.

I. Hence 'tis natural to suppose, that equal Lines moving thro' the same or equal Spaces, will describe equal Surfaces.

II. Equal Lines, with uniform or equable Motions (*i. e.* being neither accelerated nor retarded) in equal Times, will describe equal Surfaces: And if they do thus describe equal Surfaces, it must be in equal Times.

III. Hence

III. Hence also, if the Line a in a given Time describe the Parallelogram A , and the equal Line b



in the same Time describe the Oblique Parallelogram B or C , whose Perpendicular Altitude is the same with that of A : Those Parallelograms will be all three equal one to another.

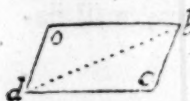
Because the Oblique Motion, which the Line b hath, whereby 'tis carried either to the right or left Hand, is by no means contrary to the direct Motion downward; and consequently, the Line b will move the same perpendicular Distance in the same Time, with an equable Motion, whether the latter Motion be impressed upon it or not. Wherefore,

IV. All Parallelogrammick Figures, with equal Bases and equal Perpendicular Altitudes, must be equal.

6. In every Parallelogram, the opposite Angles are equal. Let the Parallelogram be



oc : I say, the Angle o is equal to c ; for the Angle o is equal to the alternate one b (1. 31.) and the external one b is equal to the internal one c (1. 31.) wherefore o is equal to c .



7. A Line, as db , drawn a-cross the Figure from Angle to Angle, is called the Diagonal, and by some, the Diamerer.

8. Every Parallelogram is divided into two equal Parts by the Diagonal. The Diagonal bd divides the Parallelogram oc into the two equal Triangles obd and bcd . For, 1. The Angle o is equal to c (3. 6.)

2. The

2. The Angle obd is equal to $cd b$ (1. 31.) and the Side bd is common to both these Triangles; wherefore the Triangle obd is equal to cbd (by 2. 14.)

9. In every Parallelogram, the opposite Sides are always equal.

For (drawing the Diagonal db) the whole Triangle dob will be equal to the Triangle bcd , by the foregoing Prop. And consequently, the Side cd must be equal to ob , and the Side od to cb .

10. Two Diagonals, as ac and bd , do biffect each other in the Middle at e .

For in the two Triangles, aed and bec , the Side ad is equal to bc (3. 9.) The Angle ead is equal to ecb (1. 31.) and moreover the (Vertical) Angles aed and ceb are equal also (1. 23.) Wherefore the whole Triangle aed is respectively equal to the Triangle bec (2. 14.) And consequently, the Side de is equal to eb , and the Side ae to the Side ec . The two Diagonals therefore biffect each other in the middle. Q. E. D.



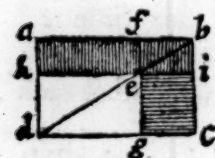
11. Every Right Line, as fg , passing through the middle of a Diagonal, divides the Parallelogram into two equal Parts.

To demonstrate which, the Trapezium or Irregular Quadrilateral Figure $fgda$ must be proved equal to the Trapezium $fgcb$. And that is thus done. 1. The Triangle bef is equal to the Triangle deg : For the Side de is equal to eb by the Supposition; and the Angle efb is equal to egd (1. 131.) and the opposite Angles at e are equal; wherefore the Triangle efb is equal to edg (2. 14.) 2. The great Triangle abd is equal to bdc (3. 8.) wherefore if from the Triangle abd you take away the little Triangle feb ,
and



and instead of it put the Triangle edg (which is equal to feb) you will have the Trapezium $fadg$, which will be equal to the Triangle adb : That is, to just one half of the whole Parallelogram (3. 8.) which was to be proved.

12. If in the Diagonal db you take a Point as e , and thro' it draw two Lines, hi and fg , parallel to the two Sides of the Parallelogram, it will be divided by them into four lesser Parallelograms, i. e. fi , bg (which two are called the Parallelograms about the Diameter) and ae , ec ; which other two are called the Complements. And those two Complements with either of the Parallelograms about the Diameter, make a Figure that is called a Gnomon. As you see in the Figure, where the Gnomon is distinguished by being shaded.



13. In every Parallelogram the Complements are equal. We must prove that ea is equal to ec .

DEMONSTRATION.

The whole Triangle abd is equal to the whole bdc (3. 8.) and the little Triangle efb is (for the same Reason) equal to ebi . And the Triangle bed is also (by the same) equal to edg . Wherefore if from the two equal Triangles, abd and bdc , we take away equal things, viz. if from one we take away efb and dhe , and from the other ebi and egd , there will remain on one Side the Parallelogram ea , equal to the Parallelogram ec , which remains on the other; which was to be proved.

14. Parallelograms having the same Base, and being between the same Parallels, are equal.

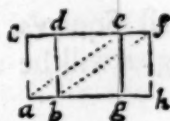
Let there be a Parallelogram bc , and another af , both on the same Base ab ; and let the Line cd , when produced, be supposed to pass by ef ; so that the two Parallelograms shall be between the same Parallels, and terminated by them; that is, between the two Parallels cf and ab . I say then, that the Parallelogram cb is equal to af .



For ca is equal to bd , and ae equal to bf , because opposite Sides of Parallelograms, and the Angles at c and d equal (by 29. 1.) wherefore the Triangle cae is equal to the Triangle dbf . Now if from each of these equal Triangles be taken the little Triangle doe , and to the Remainders be added the Triangle aoe , the Parallelogram ad will be equal to the Parallelogram af . Q. E. D.

15. Parallelograms on equal Bases, ab and gh , and between the same Parallels ab and cf , are equal.

For if we imagine the third Parallelogram fa to be drawn, that shall be equal to the Parallelogram ad , because on the same Base ab with it, and between the same Parallel Lines ab and cf . And that Parallelogram will also be equal to eh , because it hath the same Base ef with it (it matters not whether you reckon the Base above or below) and it is between the same Parallels. Therefore he and bc , being both equal to the third Parallelogram fa , must be equal to each other.



16. Triangles on the same Base ab , and being between the same Parallels, cf and ab , are always equal.

The Triangle abc is equal to aeb : Because if you imagine a Line bd drawn parallel to ac , and another, as df , drawn parallel to ae ; there will be made two Parallelo-

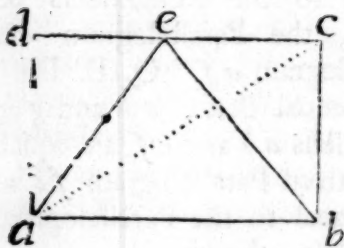


Parallelograms $acdb$, and $aefb$; which being on the same Base, and between the same Parallels, will be equal to one another. (3. 14.)

But the Triangle abc is the half of the Parallelogram $acdb$, and the Triangle abe is the half of the Parallelogram $aefb$ (3. 8.) wherefore (*since the Wholes are equal, the Halves must*) and consequently the Triangle acb is equal to the Triangle aeb .

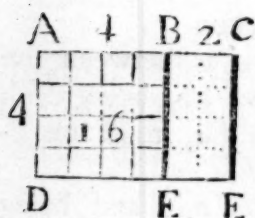
17. Triangles on *equal Bases*, and between the same Parallels, are also equal; as is very easy to prove from (3. 15.)

18. If a Triangle acb have the same Base with a



Parallelogram, and be also between the same Parallels, it shall be just the half of that Parallelogram. For it will still be equal to abc , which is just half (3. 8.) of $abcd$.

The Mensuration of all Squares, Rectangles, Parallelograms and Triangles will be understood from what hath been deliver'd above; If you suppose,



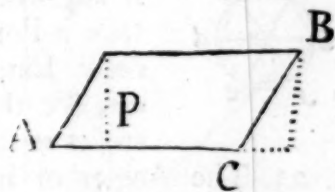
1. That the Describent AB or AC , before its Motion, be divided into any determinate Number of equal Parts; and the Dirigent (now supposed to stand at Right Angles with it) into the same or any other Number of such Parts; for then the

Motion of the Describent Line, thus mark'd out by Points into Units, will describe a Square (if the Dirigent be equal to it) and a Rectangle, if it be unequal. Which Square or Rectangle will be divided into as many little Squares as there are Units in the

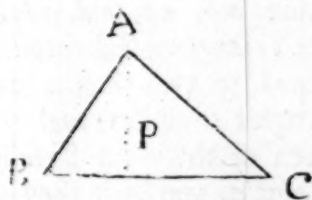
the Product of the Number of the Divisions, or equal Parts in one Line, multiplied by those in the other; That is, $AB\ 4$ multiplied by $AD\ 4$, produces 16 , the Square of 4 . And $AC\ 6$ multiplied by $AD\ 4$, produces 24 , the Rectangle under AC and AD . So that what is a Product in Numbers or in Arithmetick, in Lines, or in Geometry, is called a Rectangle. And therefore you will find the Latin Writers of Geometry, when AC is to be multiplied by AD , not saying *Multiplica*, but *Duc AC in AD*: That is, carry the Line AC along the Dirigent AD , in a Normal Position to it, 'till it come to end, and then it will form the Rectangle $AF=24$; wherefore the Area of a Square is found, by multiplying the Side AB into itself.

The Area of any Rectangle, as AF , is found by multiplying the Side AC by AD .

And since a Rectangle on the same Base and of the same Altitude with a Parallelogram is equal to it; to find the Area of any Parallelogram, as AB , you must multiply the Side AC by a Perpendicular, as P , let fall from the other Side to it.



And since every right-lined Triangle is the half of a Parallelogram or Rectangle of the same Base and Altitude: To find the Area of the Triangle ABC , you must multiply any Side, as BC , by a Perpendicular, as P , let fall to it from an opposite Angle, and take half the Product: Or if either P or B happen to be even Numbers, multiply one



one by $\frac{1}{2}$ of the other, the Product is the Area of the Triangle.

19. A *Pentagon* is a Figure having five Sides and five Angles.

If all the Sides are equal, and consequently the Angles, 'tis called a *Regular Pentagon*.

20. An *Hexagon* is a Figure of six Sides and Angles, an *Heptagon* of seven, an *Octagon* of eight, &c. which are all called *Regular* when they have equal Sides and Angles.

21. A *Polygon* in general signifies any Figure of many Sides and Angles; but no Figure is called by this Name, unless it have more than four or five Sides.



22. Every *Polygon* may be divided into as many Triangles as it hath Sides, if any where within the *Polygon* you take a Point, as *a*, and from thence draw Lines to every Angle *a b*, *a c*, *a d*, &c. they shall make as many Triangles as the Figure hath Sides.

23. The Angles of any *Polygon* taken all together, will make twice as many Right ones, except four, as the Figure hath Sides, *v. gr.* If the *Polygon* have six Sides, the double of that is 12; from whence take four, there remains eight. I say, that all the Angles of that *Polygon*, viz. *b, c, d, e, f, g*, taken together, are equal to eight Right Angles. For the Lines *a b*, *a c*, *a d*, &c. do divide the Figure into six *Triangles*; the three Angles of each of which are equal to two Right ones (2. 9.) so that all their Angles together make 12 Right ones. But now, each of these six *Triangles* hath one Angle in the Point *a*, and by it they compleat the Space all round the said Point. And all the Angles about that Point, are equal to four Right ones (1. 12.) Wherefore those four being taken from 12 (*the Sum of the Right*

Right Angles of all the six Triangles) leaves eight, the Sum of the Right Angles of the *Hexagon*, which makes 8 times 90, or 720 Degrees; and therefore each Angle must be $\frac{1}{2}$ of that, viz. 120 Degrees.

So that the Figure hath plainly twice as many Right Angles as it hath Sides, except four; which was to be proved.

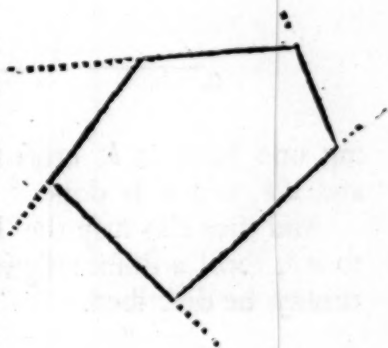
C O R O L L A R Y.

All the external Angles of any Right-lined Figure, are equal to just four

Right Ones: For drawing out the Sides, as in the Figure, 'tis plain the internal and external Angles together will make twice as many Right Ones as the Figure hath Sides; but the internal Angles are equal to all those, except four (by this *Prop.*)

Wherefore the external Angles must make up these four, and no more.

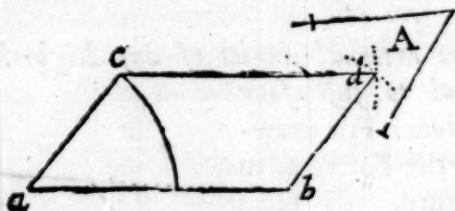
24. A *Polygon* may be divided also into *Triangles*, by drawing Lines from Angle to Angle. But then the Number of the Sides will exceed that of the *Triangles*. And hence the *Area* of any Right-lined Figure may be found, by reducing it into *Triangles*, and then finding the *Area* of each *Triangle* severally, adding all into one Sum.



PROBLEM I.

On a Line given $a b$, to make a Parallelogram, having an Angle equal to a given Angle A .

Make the Angle $c a b = A$. Then take $a b$ in your Compasses, and setting one Foot in c , strike an Ark, as d : Next take the Distance $a c$, and plac-



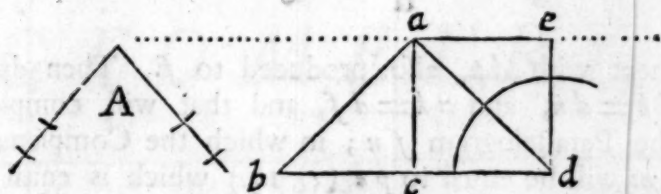
ing one Foot in b , cross the Ark in d ; draw $c d$ and $d b$, and it is done.

And thus also may the Line $c d$ be drawn parallel to $a b$, thro' a Point assigned, and any Parallelogram readily be described.

PROBLEM II.

A Triangle $a b d$ being given, to make a Parallelogram equal to it, which shall have a given Angle equal to A .

Bisect the Base of the Triangle in c : Make the Angle $c d e = A$, thro' the Vertex a draw $a e$ paral-



lel to the Base $b d$. Make $a e = c d$, and draw $a c$. So will $c e$ be the Parallelogram required.

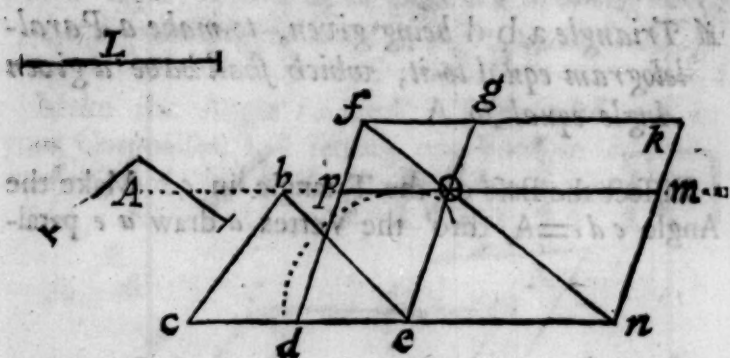
For being on but half the Base, and of the same Height with the Triangle, it will be equal to it, by the 18th of this Book, and its Angle $c d e$ is equal to A . Q. E. F.

PROBLEM III.

On a Line given, as L , to make a Parallelogram equal to a given Triangle $c b e$, and having an Angle equal to an Angle given, as A .

Make the Parallelogram $d o$ equal to the Triangle, and having its Angle $e = A$, by Problem the last.

last. Then produce po , 'till om become equal to L , and draw out de , 'till en be also equal to L . Then draw the Diagonal no , producing it 'till it



meet with dp , also produced to f . Then draw $fk = dn$, and $nk = df$, and that will complete the Parallelogram fn ; in which the Complement gm will be equal to pe (3. 13.) which is equal to the Triangle cbe . Q. E. F.

And thus 'tis easy to make a Parallelogram equal to any Right-lined Figure given, by reducing that Figure into Triangles, &c.





ELEMENTS

OF

GEOMETRY.

BOOK IV.

Of a Circle.

1.  Line is said to *Touch* (or to be a *Tangent* to) a Circle, when, though produced both Ways from the Point of Contact, it will only touch it, and not cut or enter within it. Thus the Line *a* touches the Circle *C*, as that Circle *C* doth the Circle *D*, but *d* enters within the Circle, and cuts it, and is called a *Secant*.



2. If a Right Line enter within a Circle, and cut it into two Parts, those Parts are called *Segments*: *b* is a less Segment, and *D* a greater: That Part of

the Line cutting the Circle (*and which is within it*) is called a *Chord*, as *ef*. And the Parts of the Circle (*or rather Circumference*) cut off, are called *Arks*: The Chord with the Ark makes two mix'd Angles, as *e* and *f*, and they are called *Angles of a Segment*.

3. If you take a Point, as *c*, in the Ark of any Segment, and from thence draw two Lines *ca* and *cb* (to the Ends of the Chord) they shall make an Angle *acb*; which is call'd an *Angle in a Segment*: And that Angle *acb* is said to *infixt* or *stand* on *abd*, the Ark of the other Segment below.



4. A *Sector* of a Circle is a mix'd Triangle comprehended between two Radii, *ab*, *ac*, and the Ark of the Circle *bc*; 'tis mark'd in the Figure by being shaded.

5. If at the End of any Radius, or Semidiameter, *ab*, you draw a Perpendicular, as *db*, it shall touch the Circle but in one Point. And all the



Points of the Line *bd* shall be without the Circle; *v g.* I say, the Point *d* (or any other assignable) is without: For if you draw the Line *ad* from the Center, and that shall cut the Circle in the Point *c*, that Line *ad* will be longer than *ab*; (2. 17.) and consequently longer than *ac*, which is equal to *ab* (1. 14.) Wherefore the Point *d* is without the Circle. Q. E. D.

6. A *Chord*, as *bc*, is divided into two equal Parts (or bisected) by a Perpendicular *da*, drawn from the Center *a*. For the Triangle *abc* is an *Isoceles*, because *ba* is equal to *ca* (1. 14.) and therefore the Perpendicular *ad* bisects the Base *bc* (2. 16.) The Ark *bc* is also by this means bisected.



7. Two

7. Two Tangents, cb and cd , drawn from the same Point without a Circle, are equal one to another. For, draw from the Center to the Points of Contact, ba and ad ; then will those Lines be Perpendiculars to the Tangents (by 4. 5.) Then if you draw also the Line bd , the Angle abd will be equal to adb (2. 15.) Wherefore if from the Right, and (consequently) equal Angles cba and cda , you take away the equal one abd and adb , the remaining Angles cbd and cdb will be equal: Wherefore their opposite Sides must also be equal by the *Converse of* (2. 15.) That is, cb is equal to cd . Q. E. D.



8. Equal Chords, as bc and fh , do cut off equal Segments bdc and fgh . And the Perpendiculars ae and ai , drawn to them from the Center, are also equal, as is easily proved; (saith *Pardie*, but he gives us no *Demonstration*.) Yet tis plainly thus proved; The Chords and Arks are both bisected by the Perpendiculars (4. 6.) And therefore the Sectors cad , dab , fag and gah , must be all equal; as also will all the Triangles x , z , o and k , by (2. 11.) Therefore their Doubles will also be equal, i. e. the Sector bac will be equal to fah : And the Triangle bac to the Triangle fah . And if these last Triangles are taken from the equal Sectors haf and bac , the Segments bcd and hgf must remain equal. That the Perpendiculars are equal, is plain from the Equalities of the Triangles z and o , or X and K .



9. Let there be a Semidiameter Rc , and a Perpendicular (to it without the Circle) RT , another Line cutting the Circle in S , and a Perpendicular (let fall from thence) to the Radius RC in n (a Point within the Circle.) All these Lines have Artifi-



cial Names. The Line TR is called the *Tangent* of the Ark RS (*which suppose 30°*) TC is called the *Secant* of the same Ark of 30° , and the Line S is *called* the (Right) *Sine* of the same Ark. RC is by some called the whole Sine, but most usually the *Radius*. And VR is called the *Verfed Sine* of the same Ark.

10. If in the Circumference of a Circle, you take two Points, as a and b , and from thence draw two Lines to the Center c , and two others to any Point, as d , in the Circumference; they will make two Angles, of which acb is called an *Angle at the Center*, and adb an *Angle at the Circumference*.



11. The Angle at the Center acb is always double to one at the Circumference adb (*insisting with it on the same Ark $a b$* .)



Of which there are three Cases.

I. If one of the Lines, as db , pass thro' the Center c , then 'tis plain the external Angle acb (2. 10.) will be equal to both the internal and opposite ones a and d taken together.

But the two Angles d and a are equal, because acd is an *Isoceles Triangle*, whose Side ac is equal to cd (2. 15.) Therefore the Angle c at the Center being equal to both, is double of either alone: That is, double to d . Q. E. D.

II. If neither of the Lines db , de (*which form the Angle at the Circumference*) pass thro' the Center c : (*but fall both on the same Side of the Diameter*) Let the Diameter dce be drawn. Then will the whole Angle ace (*at the Center*) be double to the Angle ade (*at the Circumference*)



cumference) by what was proved in the first Case. Also the Angle bce is double to bde , by the same. Wherefore if from the Angle ace , we take away that bce , and from the Angle ade , which is the half of ace , we take away bde , which also is the half of bce , the remaining Angle adb must be just the half of acb . For 'tis as plain as an Axiom, that if one Quantity be double to another, and you take away from the Bigger, just the Double of what you take from the other, the Remainder of the Bigger must be double to the Remainder of the Lesser.

III. If the Diameter fall between the Lines forming the Angle at the Circumference: Then will, as before, the Angle ace be double to abe (by Case 1. of this) and the Angle ecd will be double to ebd by the same; therefore the whole Angle acd must be double to abd . (So that in all Cases the Angle at the Center is double to one at the Circumference, if they both stand on the same Ark, or (which is all one) are in the same Segment.)



12. All Angles (in the same Segment or) insisting on the same Ark ab , are equal, let them terminate in any Part of the Circumference whatsoever.



For the Angle adb will be equal to aeb , because each is the half of the Angle at the Center acb (4. 11.)



13. An Angle at the Center bce , standing on half of the Ark aeb , is equal to the Angle adb at the Circumference, standing on the whole Ark; for c is equal to twice x ; (by 4. 11.) and x is equal to o , that is, to half abd , (4. 6, and 4. 8.) Wherefore c is equal to adb . Q E D.

14. The

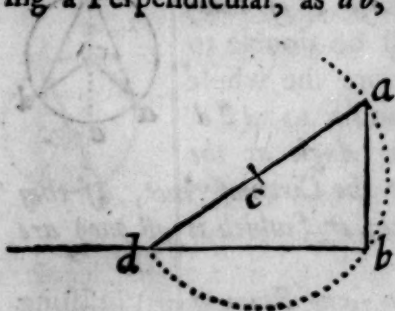
14. The Angle adb standing on the Semi-circumference aeb (or being in the Semi-circle adb) is a Right One. Let ce be drawn bisecting the Semi-circumference aeb ; then is (by the *Precedent*) the Angle ace at the Center, standing on half a Semi-circle (or on a *Quadrant*) equal to adb at the Circumference, which stands on twice that Ark, or on a Semi-circle. But ace is a Right Angle; wherefore adb (its equal) must be so too.



COROLLARY I.

Hence is derived the common Practice of Erecting a Perpendicular, as ab , at the End of a given Line. For opening the Compasses to any convenient Distance, set one Point in c , and with the other draw the Ark dba , cutting the given Line in d ; then a Ruler laid from d to c shall

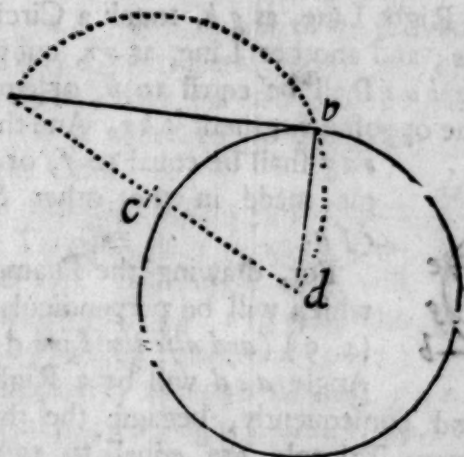
find the Point a , which is perpendicularly over b : For the Angle dba , being in a Semi-circle, is a Right One.



COROL.

COROL. II.

Hence also arises this expeditious Practice of drawing from a Point given, as a , a Tangent, as ab , to a given circle. For joining the Points a and d ,



the Center of the Circle, bisect their Distance ad in the Point c : On c , as a Center, describe the Semi-circle abd : So shall ab be a true Tangent, because the Angle abd being in a Semi-circle, is a Right One.

15. The Angle abd in a Segment less (than a Semi-circle) is Obtuse: Because the Ark aed being more than half the Circumference, its half, the Ark ae , must be more than 90° ; therefore the Angle abd , which is equal to ace (4. 13.) must also be more than 90° , that is, obtuse.



16. The

16. The Angle abd made in a Segment greater than a Semicircle, is *Acute*.



For 'tis equal to the Angle ace (4. 13.) whose Measure ae being the half of aed , an Ark less than a Semicircle, must be less than 90° . And therefore abd is less than 90° . (*i. e.*) *Acute*.

17. If a Right Line, as gb , touch a Circle, as in the Point a ; and another Line, as ae , cut it there; The Angle bae shall be equal to b , or any Angle made in the opposite Segment abe . And the Angle



eag shall be equal to f , or any Angle made in the other Segment efa .

For, drawing the Diameter ad , which will be perpendicular to ab , (4. 9.) (and also the Line de ;) The Angle aed will be a Right One; (4. 14.) and consequently, because the three Angles of every Triangle are equal to two Right Ones, (2. 9.) the Angle ead , together with d , must make just another Right Angle.

But that Angle dae , together with eab , doth make also a Right One, because the Radius ca is perpendicular to the Tangent ab : Wherefore take away ead from both, and then eab will remain equal to d ; and consequently to b , or to any other Angle in that Segment abe , or that stands on the same Ark efa : For all those Angles are equal (by 4. 12.) The Angle eab therefore is equal to b ; which is the first Part of the Proposition.

We must next prove the Angle gae to be equal to f ; which is the other Part.

In the Triangle afe , all the three Angles e , f and a , are equal to two Right Ones (2. 9.) And the Angle c is equal to f , by the first Part of this Proposition;

position; for $f a$ may be considered as cutting the Circle in the Point a , where $a b$ touches it, and consequently $f a b$ will be equal to any Angle that can be made in the opposite Segment $a b d e f$; and therefore to e . Now the two Angles $e a f$, and $f a b$ (that is e) together with f , are equal to two Right Ones, (2. 9.) and so are $a f$, and $f a b$ taken together with $g a e$ (1. 20.) Wherefore the Angle f is equal to $g a e$. Which was to be proved.

18. Every Quadrilateral Figure, as $d e f a$, inscribed in a Circle, hath its two opposite Angles taken together (as d added to f) equal to two Right Ones.

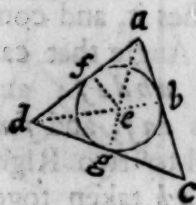
For if thro' the Point a , there be drawn a Tangent, as $g b$, and a Diagonal, as $e a$; the Angle at f will be equal to $g a e$ (4. 17.) and the Angle $e a b$ will be equal to d (4. 17.)



And consequently the two Angles, $g a e$ and $e a b$, being equal to two Right Ones (1. 20.) the Angles d and f taken together, must be so too.

After the same Manner might the other two opposite Angles, $d a f$, and $d e f$, be proved equal to two Right Ones, by drawing another Tangent thro' the Point f .

19. The Converse of this Proposition is also manifest; viz. That if any Quadrilateral Figure have its opposite Angles equal to two Right Ones, it may then be inscribed in a Circle; that is, a Circle may be made that shall touch or pass through all its four angular Points.



20. A Rectilineal Figure is said to be circumscribed about a Circle, when all its Sides touch the Circle without cutting it. Thus the Triangle dae is circumscribed about the Circle bgf ; because every Side of the Triangle touches the Circle in b , g and f .

21. A Figure is said to be *Inscribed* in a Circle, when all its Angles are in the Circumference of that Circle, as the Triangle abc , in the following Figure.

22. Every Triangle, abc , may be inscribed in a Circle; for if two Lines, as eh and ei , are drawn perpendicularly bisecting the Sides ba and cb , they will cross or meet each other in the Point e , on which, as on a Center, a Circle may be drawn which shall pass through b . And I say also, that that Circle shall pass through a and c .



For, I. The two Triangles eib and eia are equal; because ib is equal to ia by the Supposition, the Side ei is common to both, and the Angles at i are Right. Wherefore the Side eb is also equal to ea (2. 11.)

II. And for the same Reason the Triangles ehc and ehb may be proved equal, and consequently, the Side ec also will be equal to eb and to ea . But if those three Lines are all equal, the Point e , where they meet, must be the Center of a Circle of which they are Radii: And therefore the Triangle is circumscribed by a Circle. Q. E. D.

And thus may a Circle be made to pass through any three Points, if they be not all in a Right Line.

23. Every

23. Every Triangle may have a Circle inscribed in it, or be circumscribed about one. *Vid. Fig. in Art. 20.*

For drawing the Lines ae and ed , bisecting the Angles a and d , and from the Point e , where they cross, letting fall the Perpendiculars (to the Sides of the Triangle) eb , ef and eg ; I say, that if you draw a Circle on the Center e through b , that Circle shall touch all the Sides of the Triangle in the Points b , f and g .

For, I. The two Triangles $ae f$ and aeb are equal, as having the Side ae common, the Angles at f and b Right, and those at a equal (by the Supposition:) Wherefore eb is equal to ef . (2. 14.)

II. By the same Method eg may be proved equal also to ef , (that is to eb) so that these three Lines being all equal, a Circle will pass through their three Extremities, of which Circle they will be Radii; and being also all perpendicular to the Sides of the Triangle, the said Sides are Tangents to that Circle (4. 5.) and therefore do circumscribe it (by 4. 18.)

24. Every Polygon circumscribed about a Circle is equal to a Rectangled Triangle, one of whose Legs shall be the Radius of the Circle, and the other the Perimeter (or the Sum of all the Sides) of the Polygon.



Let the Line FA be equal to the Radius fb , and to it, at Right Angles, draw the infinite Line $ABCD$, &c. out of which take Ab equal to ab , bB equal to bb , Bi equal to bi , and iC equal to io , &c. So that the whole Line $ABCDEA$ may be equal to the whole Compass, or *Perimeter* of the Polygon $abcdea$. Also draw FF parallel to AA , so that all the Perpendiculars, Fb , Fi , Fk , &c. may be equal to the Radius fb , or fi , &c. 'Tis then plain, that the Triangle AFB will be equal to the Triangle afb in the Polygon, and the Triangle BFC , to bfc ; and also CFD to efd , &c. So that all these Triangles taken together, will be equal to all these in the Polygon, or to the whole Polygon.

But the Triangle FAA is equal to all the five Triangles within the Parallels; because drawing the Lines BF , CF , DF , &c. the Triangle FAB will be equal to FAB , FBC to FBC , &c. (3. 16.) Wherefore the Triangle FAA is equal to the Polygon, which was to be proved.

25. Every regular Polygon is equal to a Rectangled Triangle, one of whose Legs is the Perimeter of the Polygon, and the other a Perpendicular drawn from the Center to one of the Sides of the Polygon. The Proof of which is the same as that in the precedent Proposition: For all the Perpendiculars fb , fi , fk , &c. are equal, &c. See the last Figure.

Wherefore the Area of every regular Polygon is found by multiplying the Perpendicular let fall from the Center of the inscribed Circle by any one Side; and then multiplying the half of the Product by the Number of the Sides.

26. Every Polygon circumscribed about a Circle, is bigger than it; and every Polygon inscribed, is less than the Circle; as is manifest, because the thing containing, is always greater than the thing contained.

27. The *Perimeter*, or (as some call it, tho' improperly) the Circumference of every Polygon circumscribed about a Circle is greater than the Circumference of that Circle; and the Perimeter of every Polygon inscribed, is less.

28. If in any little Segment of a Circle, you inscribe an Isosceles Triangle, as abc ; so that ab be equal to bc ; I say, that Triangle shall be greater than half that Segment. For if you draw a Tangent ebd , which shall be parallel to ca ; and which shall be, as ca is, perpendicular to the Radius bf ; (4. 5.) (4. 6.) And then compleat the Rectangle $adec$; that Rectangle will be greater than the whole Segment acb : But the Triangle abc , is the half of that Parallelogram (3. 18.) And therefore must be greater than half the Segment abc .



29. Let there be a Tangent adb , a Secant $fc b$, a Chord ac , and another Tangent cd ; I say, that the Triangle dbc is more than half the mixt Triangle acb , comprehended between the Lines ab , bc , and the Ark of the Circle ac . For in the Triangle dbc , the Angle c , being a Right one (4. 5.) the Side db , is longer than dc (2. 17.) That is, than da ; which is equal to dc (4. 7.) wherefore the Triangle bdc (having a longer Base, but the same Height with adc) must be greater than it; as may be collected from (3. 7.) And therefore it must be greater than the



half of the whole Triangle acb . But the Triangle acb , is greater than the mixt Triangle, made by the Ark ac , and the Right Lines, ab and ac ; and therefore the Triangle bdc , (which is more than half of acb) must be greater than the half of the mixt Triangle abc . Q. E. D.

30. From these two last Positions, it follows, that by multiplying the Sides of Polygons, you may make them so *circumscribed* about, or *inscribed* in Circles, that the Difference by which the circumscribed exceeds, or the inscribed wants of the Circle, shall be as small as you will: Because if from any Quantity whatever, you take more than the half, and from the Remainder more than its half, and again from that Remainder more than its half; you may by doing this very often, at last come to leave a Remainder as small as you please; as is self-evident. Thus (*See the 28th Figure*) after a Triangle is inscribed in a Circle that shall be less than it by the three great Segments, you may inscribe an Hexagon that shall exceed the Triangle by those three Segments, but shall be less than the Circle, by the six little Segments that are left white in the Figure.

But those six white Segments taken together, do not contain so much Space as the half of the three former shaded ones, (4. 28.) After this you may also inscribe a *Duodecagon*, which will be lesser than the Circle by 12 smaller Segments; which 12 Segments will still be less than the half of the six Segments of the Hexagon: And thus may you, by increasing the Number of Sides of the Polygon, lessen the Difference by which the circumscribing Circle exceeds it, as much as you please. So likewise on the other Hand, you might have first circumscribed a Triangle, then an Hexagon, and then a Duodecagon, &c. (*and have made, that way, the Difference between the circumscribing Polygon and the Circle, as small as you would.*)

31. Every

31. Every Circle is equal to a Rectangled Triangle, one of whose Legs is the Radius, and the other a Right Line equal to the Circumference of the Circle. For such a Triangle will be greater than any Polygon inscribed, and less than any Polygon circumscribed, (by 24, 25, 26, and 27 of this fourth Book.) And therefore must be equal to the Circle.

For should it be greater than the Circle, be the Excess as little as it will, a Polygon may be circumscribed, whose Difference from the Circle shall be yet less than the Difference between that Circle and the Rectangled Triangle, and that Polygon will be less than the Triangle, which is absurd. And if it be said that this Rectangled Triangle is less than the Circle, an inscribed Polygon may be made, which shall be greater than that Triangle, which is impossible.

" This kind of Demonstration, which we here
 " use, and which is called *Reductio ad Absurdum*, sive
 " *ad Impossibile*, is one of the finest Inventions of the
 " Ancients: And on it is founded all the *Geometry*
 " of *Indivisibles*; so that I cannot but much wonder
 " some of our Modern Authors should reject it as
 " indirect and deficient. But if we must arrive to
 " such a Point of Niceness, that we can't bear any
 " Demonstration, unless it be direct and Positive;
 " 'tis easy enough to give this before us such a Turn,
 " as shall render it Regular and Direct.

" For this cannot but be admitted as a Principle;
 " That if two determinate Quantities a and b are such,
 " that every other imaginable Quantity, which is
 " greater or less than a , is also greater or less than b ;
 " these two Quantities a and b must be equal. And
 " this Principle being granted, which is in a manner
 " self-evident, it may directly be proved that the
 " Triangle (before-mention'd) is equal to the Circle:
 " Because every imaginable inscribed Figure, which

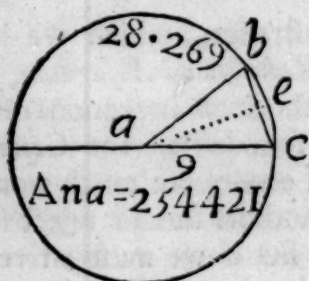
“ is less than the Circle, is also less than the Triangle: And every circumscribed Figure greater than the Circle, is also greater than the Triangle.”

This is that which is called the Quadrature of (or squaring) the Circle, which consists in finding a Square, Triangle, or any other Rectilineal Figure exactly equal to a Circle. And this would easily be done, could we find a Right Line equal to the Circumference; as is plain from this last Proposition. But such an Equality is not to be found Geometrically.

To find the Area of a Circle.

Since the Circle is equal to a Right-angled Triangle, whose Base is the Radius, and the Perpendicular a Line equal to the

Circumference; half the Product of the Radius into the Periphery, will give the Area of the Circle.



In Practice, therefore say, either as 7 : to 22 :: So is the Diameter in Inches, equal Parts, &c.

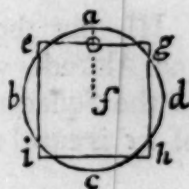
to the Circumference, or more nearly and without Division, say, as 1000 is to 3141 :: So is the Radius of any Circle in Inches (suppose 9 Inches) to 28 269, which therefore will be Semi-circumference: And this multiply'd by 9 Radius, gives 254.421, for the Area required.

For the Area of a Sector or Segment of any Circle.

Since a Circle may be conceiv'd as an Aggregate of an infinite Number of Isosceles Triangles, whose common Vertex is the Center; any Portion of the Periphery, as *bc*, being considered as a strait Line, and

and the Perpendicular ae let fall, the Area of the Sector must be half the Product of the Ark bc into the Radius ae ; and if from the Sector you take the Area of the Right-lined Triangle abc , there will remain the Area of the Segment bec .

32. If a Right Line could be disposed into the Form of the Circumference of a Circle, it would contain more Space than any other Figure, or Regular Polygon whatsoever: Suppose the Circumference of the Circle, $abcd$, to be disposed into the Form of a Square, or into any other Regular Polygon: So that all the Sides eg , gh , hi , and ie together may be equal to the Circumference $abcd$; I say, the Circle is greater than that Square. For the Circle is equal to a Rectangled Triangle, one of whose Legs is the Radius fa , and the other the Circumference. And the Polygon is equal also to such a Triangle, one of whose Legs is the same Circumference $abchd$, or the Sum of the Sides $geih$; and the other Leg is the Line fo (4. 25.) But as the Line fo is less than the Radius fa , so the second Triangle, which is equal to the Polygon, must be less than the first, which is equal to the Circle; and therefore the Square or Polygon must be less than the Circle; which was to be demonstrated.



“ And this is what we mean, when we usually say, that of *Iso-perimetrical Figures* (or which have equal *Perimeters* or *Circumferences*) the greatest is the Circle.

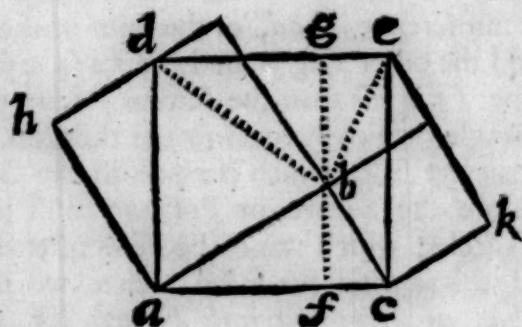
Before we go to Solids, I thought it proper to give the Learner here, this most noble Theorem of *Pythagoras*; because, tho' it be indeed demonstrated in the sixth Book, yet nearly after *Euclid's* manner, it may also be done here: Thus,

In every Right-angled Triangle as abc , the Square of the Hypotenuse ac , is equal to the Sum of the Squares of the Legs ab and bc ; For,

I. The Square of ca , is equal to the two Rectangles df and fe .

II. The Rectangle df is double of the Triangle abd , being of the same Base and Altitude; and the Rectangle fe is, for the same Reason, double of the Triangle bec . (by 3. 18.)

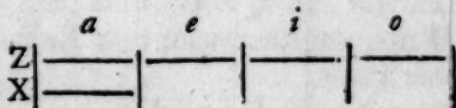
III. But those Triangles, being of the same Base and Altitude with, will be equal also to one half of the Squares bb and bk : Wherefore the Square of ac is equal to the Sum of the Squares of the Legs.



I have

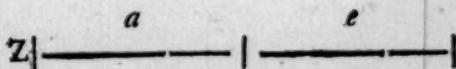
I have here added also the Substance of the second Book of *Euclid*, about the Power of Lines, &c. And I would advise the young Geometrician, before he proceeds any farther, (and if not done already) to begin the Study of Algebra; a little of which will be of excellent Use to him in facilitating the Demonstrations in Geometry, and in preparing the Mind, and enuring of it to Abstraction, before he come to the Doctrine of Proportion. And the four first Rules of *Addition*, *Subtraction*, *Multiplication*, and *Division* in *Integers* and *Fractions*, will be sufficient to enable him to understand the following Propositions: As also the most useful Ones, which he will find added (in this Edition) in all the following Books of these Elements.

I. If there be two Lines Z and X, one of which, as z, is divided into any Number of Parts, as into $a + e + i + o$. The Rectangle under the two whole Lines zx , is equal to the Sum of all the Rectangles made by x multiplied into the Parts of z .



That is, $ZX = Xa + Xe + Xi + Xo$. This is so plain, it needs no Proof.

If a Right Line, as Z, be divided into two Parts, $a + e$; the Rectangles made by the whole Line, and both its Parts, are equal to the Square of the whole Line.



That is, $za + ze = zz$.

For $za = aa + ae$.

And $ze = ae + ee$.

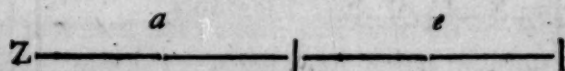
That is,

$$za + ze = aa + 2ae + ee = Q \overline{a+e}.$$

Q E. D.

III. Let the Line Z be cut into $a+e$; then shall the Rectangle under the whole Line (Z) and the Part (a) be equal to the Square of that Part a , together with the Rectangle made by the two Parts a and e .

That is, $Za = aa + ae$.

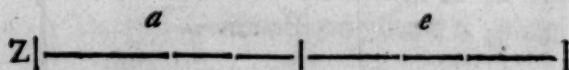


For $Z = a + e$

And $a + e \times a = aa + ae$. Q E. D.

IV. The Square of any Line, as Z, divided into any two Parts a and e , is equal to both the Squares of those Parts, together with two Rectangles made out of those Parts.

That is, $zz = aa + 2ae + ee$.



Multiply $a + e$ by itself, and the thing is plain.

$$\begin{array}{r}
 a+e \\
 a+e \\
 \hline
 aa+ae \\
 +ae+ee \\
 \hline
 aa+2ae+ee \\
 \hline
 \hline
 \end{array}$$

COROLLARY.

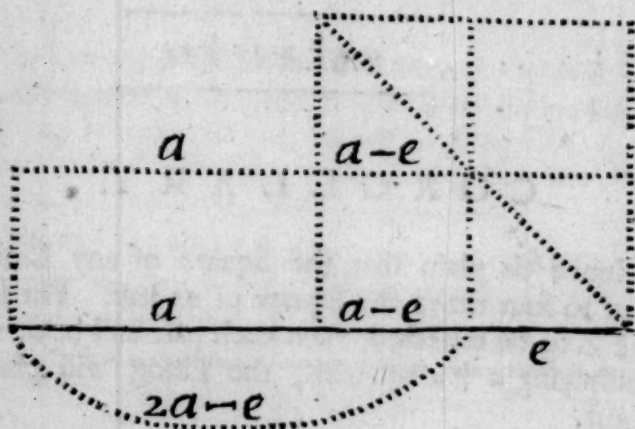
Hence 'tis plain that the Square of any Line is equal to four times the Square of its half. For suppose Z to be bisected, then each part will be a , and multiplying $a+a$ by itself, the Thing will plainly appear.

$$\begin{array}{r}
 \begin{array}{c} a \qquad \qquad a \\
 \text{Z} | \text{-----} | \text{-----} | \end{array} \\
 a+a \\
 a+a \\
 \hline
 aa+aa+aa+aa=4aa.
 \end{array}$$

V. If a Line be divided into two Parts equally, and in two other Parts unequally, the Rectangle under the unequal Parts, together with the Square of (the intermediate Part) the Difference between the equal and unequal Parts, is equal to the Square of half that Line.

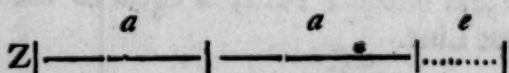
Let

Let the whole Line be $2a$, then each Part will be a . Let the lesser unequal Part be e , then the intermediate Part will be $a - e$, and the greater unequal Part will be $2a - e$; which multiplied by



e , produces $2ae - ee$; To which adding the Square of the Difference, or intermediate Part $a - e$, which is $aa - 2ae + ee$, the Sum will be only aa , the Square of half the Line.

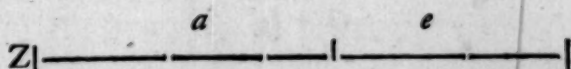
VI. If a Line be bisected, and then another Right Line be added to it, the Rectangle or Product of the whole augmented Line, multiplied by the Part added, together with the Square of the half Line, is equal to the Square of the half Line and part added, consider'd as one Line.



Let the first Line be $2a$, and the Part added e , then the whole will be $2a + e$; which multiply'd by e , produces $2ae + ee$, and the Square of half the Line

Line aa being added to it, it will be $2ae + ee + aa$, which is equal to the Square of $a + e$, by Prop. 4.

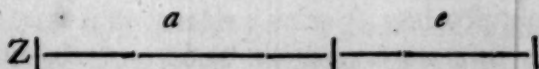
VII. If a Quantity or Line be divided any how into two Parts, the Square of the whole added to the Square of one of the Parts, shall be equal to two Rectangles contained under the whole Line, and that Part added to the Square of the other Part.



Let a be one Part, and e the other. The Square of the whole and of the lesser Part e , makes $aa + 2ae + 2ee$. Then if the whole $a + e$ be multiplied twice by e , it will produce $2ae + 2ee$; and if to this be added the Square of the other Part aa , the Sum will be

$aa + 2ae + 2ee$, equal to the former.

VIII. If a Line be cut any how into two Parts, the Quadruple Rectangle under the whole Line, and one of the Parts added to the Square of the other Part, is equal to the Square of the whole and the other Part added to it, as if it were but one Line.

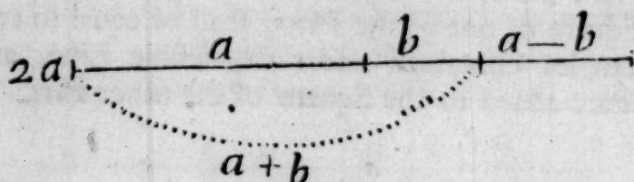


Let the whole Line be $a + e$, then four times that multiply'd by e (or the Quadruple Rectangle under that and e) will be $4ae + 4ee$; to which adding the Square of the other Part aa , the Sum will be $aa + 4ae + 4ee$.

And if you square $a + 2e$, which expresses the whole Line, with e added to it, the Product will be the former Sum of $aa + 4ae + 4ee$.

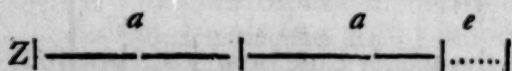
IX. If

IX. If a Line be bisected, and also cut into two other unequal Parts, the Sum of the Squares of the unequal Parts will be double to the Sum of the Squares of the half Line, and of the Difference between the two unequal Parts.



Let the whole Line be $2a$; and the Difference between the equal and unequal Parts b ; then the greater unequal Part will be $a+b$, and the lesser $a-b$: The Sum of the Squares of the unequal Parts will be $2aa + 2bb$, which is double to the Square of half the Line added to the Square of the Difference. *Q. E. D.*

X. If a Line be bisected, and then another Line added to it; the Square of the whole increased Line, together with the Square of the Part added, is double the Sum of the Squares of the half Line, and of the half Line and Part added, taken as one Line.



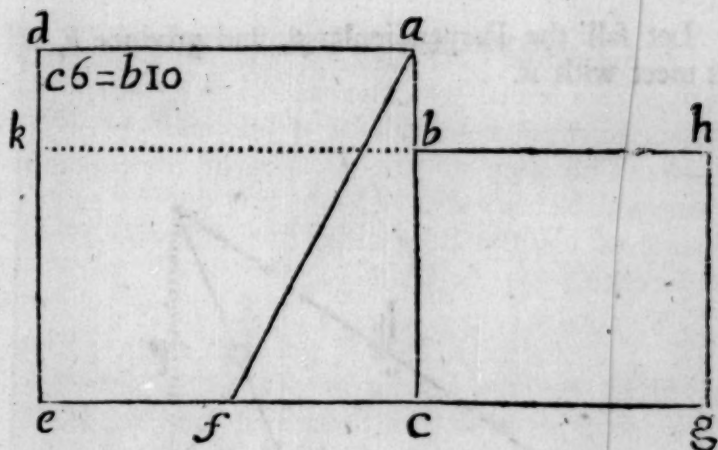
Let the whole Line be $2a$, and the Part added e ; then the whole increased Line will be $2a+e$; and the half Line and Part added will be $a+e$. The Sum of the Squares of $2a+e$, and of e , is $4aa + 4ae + 2ee$; which is plainly double to aa , and $aa + 2ae + ee$, added together. *Q. E. D.*

This

This Problem is also of frequent Use.

PROBLEM.

To divide a Line so, as that the Rectangle under the whole Line a c, and one Segment a b, shall be equal to the Square of the other b c.



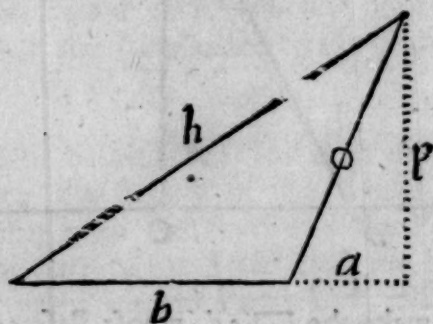
On ac make the $\square cd$, whose Base ec bissect in f , and draw af ; make $fg = af$, and compleat the $\square bg$, producing bb to k ; then is ac truly divided in b ; for the Line ec being bissected in f , and the Part cg added to it, the (by Prop. 6. of the Power of Lines) Rect. $kg + fcq = fgq = faq = acq + fcq$: Wherefore taking fcq from both, the Rect. $kg = acq$, and taking the Rect. kc from both the Rect. $db \square bg$; that is Rect. $cab = bgq$. Q. E. F.

N. B. This is called dividing a Line according to Extream and Mean Proportion; which Proportion cannot be express'd in Numbers.

PROP. I.

In an Obtuse-angled Triangle, the Square of the Side subtending the Obtuse Angle, exceeds the Sum of the Squares of the other two Sides by the double Rectangle, ($2 b a$) under the Base, and the part added to it.

Let fall the Perpendicular p , and produce b , till it meet with it.



DEMONSTRATION.

1. $hh = bb + 2ba + aa + pp$.
2. And $oo = pp + aa$.
3. But $bb + oo = bb + aa + pp$.

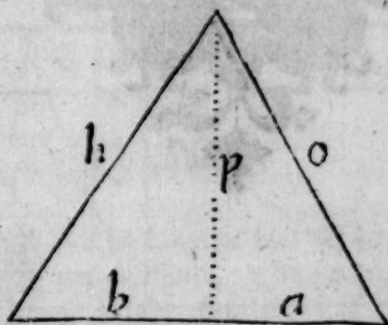
Wherefore hh exceeds the last Step by $2ba$.
Q. E. D.

PROP.

PROP. II.

In an Acute-angled Triangle, the Square of the Side (h) subtending an Acute Angle, is less than the Sum of the Squares of the other two Sides, by double the Rectangle under the whole Base, (b + a) and the Segment of the Base (a) which is next to the Acute Angle.

Let fall the Perpendicular p.



DEMONSTRATION.

1. $hh = bb + pp.$

2. $oo = pp + aa.$

3. Q. $\overline{b+a} = bb + 2ba + aa.$

4. bb

4. $bb + pp + 2aa + 2ab$, is the Sum of the Squares of the Legs.

Wherefore bb is less than that by $2aa + 2ab$, which is plainly equal to the double Rectangle under the whole Base, and the Part a .





ELEMENTS

OF

GEOMETRY.

BOOK V.

Of Solids.

1.  Right Line is said to be *Right* up-
on a Plane, when it stands on it
at Right Angles, just like a Pillar
on the Ground, and is inclined
no more to any one Side of the
Plane, than to the other.

2. Two Planes are parallel to each other, when
all the Perpendiculars that can be drawn between
them, are equal. (That is, *when they every where
are equally distant.*)

3. One Plane is right and perpendicular to another
Plain, when, like a well-made Wall, it inclines and
leans on one side no more than it does on the other.

F

4. A

4. A *Solid Angle* is made by the meeting of three or more Planes, and those joining in a Point; like the Point of a Diamond well cut.

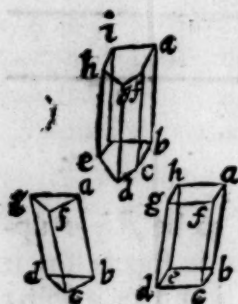


5. If we imagine a Line, as ab , fixt above in the Point a , to be moved along the Sides of any Polygon dbc ; that Line by its Motion shall describe a Figure that is called a *Pyramid*.



6. The *Polygon* is called the *Base* of the Pyramid.

7. If a Line fastened, as before, move round a Circle, as dbc , it will describe a Cone; and the Circle is its Base. And a Line drawn from the Center e to a , is call'd its *Axis*.



8. If a Line ab move uniformly about two Polygons gfa and $dc b$, which are every way equal, having their Sides and Angles mutually parallel and corresponding exactly to one another, as af to bc , fg to dc , &c. then that Line shall by its Motion describe a Figure which is call'd a *Prism*, and the Polygon is its Base.

9. If all the Sides of a Prism be a Parallelogram, then that Prism is called a Parallelopiped.

10. If a Line $a b$ move uniformly round two equal and parallel Circles, it shall describe or generate a *Cylinder*.



11. The Line joining the Centers $e e$, in the two Bases, is call'd the Axis.

“ There is no need of conceiving two Bases, equal, parallel and opposite, for the Genesis of Prisms and Cylinders. For they will be describ'd as well by imagining a Line moving round the Circumference of any plane Figure with a Motion always parallel to itself in its first Position. As if $a b$ be supposed to be carried round any of the Bases $d c b$, keeping always the same Angle with the Plane which it first had, it will describe a *Triangular, Quinquangular, or Circular Prism*, according to the Figure of the Base. And the upper end of the Line will describe a Base (as you may call it) at the Top, equal and parallel to that below.”

COROLLARY.

The Solid Content of all Isosceles Prisms and Cylinders (as also of all Parallelopipeds) is had by multiplying their Height into the Area of their Base.

And if they are scalenous Prisms or Cylinders, by multiplying the Base by the perpendicular Altitude.

But after all, this Genesis of Prisms and Pyramids of Mr. *Pardie*, respects only their Surfaces. And therefore, the most proper way to conceive the Genesis of all kinds of Prisms, is to imagine a Triangle, Quadrilateral Figure, or Polygon, or the Plane of a Circle to be moved in a Position always parallel

lel to itself; as suppose from b to e , or from g to d (in the preceding Figures) according to the Direction of the Line be or gd . Or, according to *Euclid*, a Cylinder will be generated by the Revolution of the Parallelogram $gede$ (See Fig. in Art. 8.) round about the Axis ae .

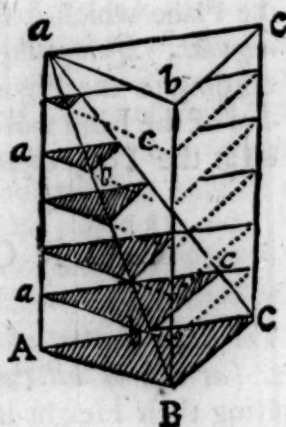
COROLLARY.

And from hence (as was observed before of Lines) 'tis plain that equal Surfaces moved uniformly over equal Places or Intervals, will describe or generate equal Solids.

And as for the Genesis of Pyramids, suppose the Triangle abc , to move downwards from the Top of a Plane Angle, determined by the two Planes aAB , aAC : Let this Motion be always parallel to itself, and let the Angular Point of the moving Triangle a , be supposed always to keep in the Line aA .

'Tis plain, as this Triangle moves farther downwards, it will still get more and more within the Solid Angle, and at last will come to be all of it within it, and to lie in the Position ABC , which will be the Base of a Triangular Pyramid, whose Vertex is at a .

The same Triangle abc , will also, by its Motion, describe another Pyramid, whose Base shall be the Parallelogram $bcBC$, and its Vertex a , as before.



13. If a Semi-circle adb be turned quite round on its Diameter ab , it will describe a Sphere or Globe, whose Axis will be ab and its Center c , the same with the Semi-circle. Every Line passing through the Center c , and terminated at each end by the Surface of the Sphere, is called a Diameter, and may be called an Axis.



14. All Lines drawn from the Center c to the Surface, are called Radii, and are all equal to one another.

To find the Surfaces of Solids.

I. For all Prisms, Parallelopipeds and Cylinders.

Find the Perimeter of the Base (which in Practice is done by girting it with a String) and multiply that by the perpendicular Height, the Product is the Surface without the Base, (*i. e.* without the top and bottom Planes) and the Bases may be found by the Rules given in *Plain Mensuration*: The Reason of which is, because a Rectangle of that Form and Dimensions will just cover the outside of the Body.

II. For Pyramids and Cones.

The Surface of a Pyramid, is only an Aggregate of Triangles, which therefore must be found severally, and then added up into one Sum.

The Surface of scalenous Cones cannot be found exactly; but for Right Ones multiply the Circumference of the Base by half of the Side of the Cone, the Product is the Area of the Convex Surface. Because the Curve Surface of a Cone is equal to a Tri-

angle, whose Base is the Periphery of the Base; and its Height the Side of the Cone; such a Figure being capable of exactly covering it.

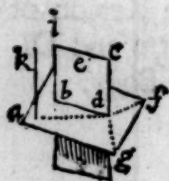
III. *For the Surface of the Sphere.*

Multiply the Diameter by the Periphery of any great Circle, or by such a Circle as hath the Diameter of the Sphere for its Diameter, the Product is the Surface. As appears from what will be prov'd below, after *Art.* 34.

IV. *The Surface of the five Regular Bodies, is easily had, by the Principles of Plain Mensuration.*

15. Two Right Lines, if they meet so as to cut or cross each other, are in the same Plane: Wherefore all the Angles and Sides of every Triangle are in the same Plane.

16. If two Planes ebd and agf cut or intersect one another, they shall do so in a Right Line, as bd ; which is call'd their common Section.



17. If a Right Line dc be perpendicular to two Lines df and dg , which are in the same Plane, that Line is also perpendicular to that Plane.

18. If a Right Line dc be perpendicular to three Right Lines df , dg and da , they are all three in the same Plane.

19. If two Lines dc , bi are perpendicular to the same Plane fga , they will be parallel to one another.

20. If two Lines dc , bi are parallel, and you draw another Line, from any Point in one to the other, as bd , those three will be all in the same Plane.

21. If two Lines dc , bi are parallel to a third ak , though that third Line be not in the same Plane with them, yet they shall be parallel to each other.

22. If a right Line ab be perpendicular to (or make any other equal Angles with) two Planes fe and cd , those Planes are parallel.

23. If two parallel Planes, dhg and afe , are cut by a third iii , the common Sections fe and hg are parallel.

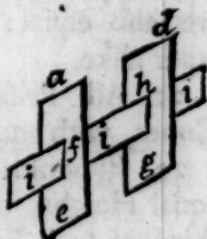
24. If a Solid Angle be made by three Plane Angles, any two of those are always greater than the third.

All these Propositions are so manifest to one that will but consider them with a little Attention, that 'tis needless to stay to demonstrate them: (And indeed the Solemn and Regular Demonstration of a Thing plain in itself, always makes it more obscure.

25. The Plain Angles, concurring to make a Solid one, taken all together, are always less than four Right Ones. For if they should make four Right Angles, they would form a Plane and not an Angle. Wherefore, that they may make a Solid Angle, they must be less than four Right Ones.

'Tis a very good way in order to gain a clear Idea of Solids and their Angles, to make the Regular Bodies out of thick Paper or Paste-board, and after the Description of every Body, you will see the Figure, which being folded up together, will express the Solid.

26. In all Parallelopipeds, the opposite Planes are equal; as is easy to conceive (from 5. 9.)



27. All Parallelopipeds having equal Bases (and Heights) or being between the same Parallels, are equal, for they are equal Aggregates of equal Parallelograms. (3. 14.)

28. Every Parallelopiped is divided into two equal Triangular Prisms, by a Diagonal Plane, which is perpendicular to its Base: For every Parallelogram of which the Figure is composed, is equally bisected.

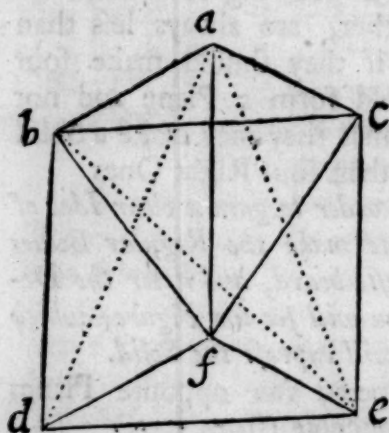
29. Triangular Prisms, having equal Bases (and Heights) or being between the same Parallels, are equal, for they are equal Aggregates of equal Triangles.

30. Pyramids having equal Bases and Heights, are also equal: For they are all supposed to grow taper alike.

31. All Prisms in general, all Cylinders and Cones, with equal Bases and Heights, are equal.

32. Pyramids and Cones on equal Bases, and of equal Heights with Prisms and Cylinders, are one third of such Prisms and Cylinders.

In a Triangular Prism and Pyramid of the same Base and Altitude, it is thus prov'd.



the whole Prism is divided into three equal Pyramids.

And

The Quadrangular Pyramid $acefb$ is divided into two equal Triangular Ones, by the Triangular Plane fbc , and the Pyramid $fcab$, is the very same with $bacf$; and this is equal to the Pyramid $dfea$: As having an equal Base and the same Altitude with it, and therefore

And since all Multangular Prisms can be divided into Triangular Ones, and that a Cylinder is only a Multangular Prism of infinite Sides, the Proposition is universally true; *That Pyramids and Cones, &c.*

N. B. A Piece of Cork or Wood, in the Form of a Triangular Prism, may be cut into three equal Pyramids.

COROLLARY I.

Hence the way of finding the Solidity of a Pyramid or Cone is discover'd, viz. To multiply the Base by $\frac{1}{3}$ of the Perpendicular Altitude.

33. Every Sphere is equal to a Cone whose perpendicular Axis is the Radius of the Sphere, and its Base a Plane, equal to all the Convex Surface of it.

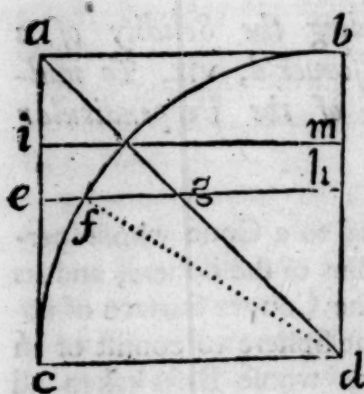
For you may conceive the Sphere to consist of an infinite Number of Cones, whose Bases taken all together compose the Surface, and whose Vertexes meet all together in the Center of the Sphere: Just as a Circle may be imagined to be composed of an infinite Number of Isosceles Triangles, the Aggregate of whose Bases makes the Circumference, and their common Vertex is at the Center.

COROL.

COROL. II.

Hence the Solidity of the Sphere will be gain'd by multiplying its Surface by $\frac{2}{3}$ of its Radius.

Let the Square $a d$, the Quadrant $c b d$, and the Right-angled Triangle $a b d$, be supposed all three to



revolve round the Line $b d$ as and Axis: Then will the Square generate a Cylinder, the Quadrant an Hemisphere, and the Triangle a Cone, all of the same Base and Altitude.

I. Then the Square of $e b$ (which is equal to the Square of $f d$, which is equal to the Square of $f b$, together with that of $b d$ (or its equal $g b$) will be equal to the Square of $g b$ ($= b d$) together with the Squares of $f b$. And since Circles are as the Squares of their Diameters (which must be now taken for granted, but will be proved in the sixth Book) the Circle made by the Revolution of $e b$, must be equal to the two Circles made by the Motion of $f b$ and $b g$. Wherefore,

II. If you take the Circle made by the Revolution of $f b$ from both, there will remain the Circle made by the Motion of $g b$ equal to the Ring describ'd by the Motion of $e f$: And thus it must always be, wherever you draw the Line $e b$, or $i m$, &c.

III. There-

III. Therefore the Aggregate of all the Rings made by the Revolution of the ef 's, must be equal to that of all the Circles made, by the Motion of the gb 's: (*i. e.*) the Dish-like Solid, formed by the Revolving Rings, will be equal to the Cone formed by the Revolution of the gb 's, which are the Elements of the Triangle abd . That is, the Dish-like Solid will be as the Cone is $\frac{1}{3}$ of the circumscribing Cylinder, and consequently the Hemisphere must be $\frac{2}{3}$ of it: Wherefore the Sphere is $\frac{2}{3}$ of the circumscribing Cylinder.

IV. Let then the Radius of the Sphere be $r = cd = bd$, then the Diameter will be $2r$; let the Surface of the Sphere generated by the revolving Semi-circle be called S ; and that of the Cylinder, formed by the Revolution of $2ac = 2r =$ Diameter, be called f . Wherefore in what was just now proved (by *Art.* 33. of this Book) the Expression for the Solidity of the Sphere in this Notation will be $\frac{rS}{3}$ —and putting c equal to the Circumference of the Base, or for the Periphery of a great Circle of the Sphere, the Curve Surface of the Cylinder (by multiplying the Altitude into the Periphery of the Base) will be $2rc$; also $\frac{rc}{2}$ will be the Area of a great Circle (by *Prop.* 26 of *Book* 4.) and this multiplied by $2r$, makes $\frac{2rrc}{2}$, which is the Solidity of the Cylinder (by *Cor. Art.* 11.) Now since f was put equal to $2rc =$ to the Curve Surface of the Cylinder $\frac{fr}{2}$ (by substituting f for $2rc$) will be also $=$ to the Solidity of the Cylinder. Now since the Sphere is $= \frac{2}{3}$ of the Cylinder, $\frac{rS}{3} = \frac{2}{3}$

$= \frac{2}{3} \frac{rf}{2}$ That is $\frac{rS}{3} = \frac{2rf}{6} = \frac{rf}{3}$. Wherefore $rS = rf$; that is, dividing by r , $S = f$, or the Surface of the Sphere is equal to the Curve Surface of the Cylinder: But the Curve Surface of the Cylinder was $2rc$.

Wherefore, to find the Area of the Surface of either Sphere or Cylinder, you must multiply the Diameter ($= 2r$) by the Circumference of a great Circle of the Sphere, or by the Periphery of the Base. From this Notation also $\frac{rc}{2}$, the Area of a great Circle of the Sphere, is plainly $\frac{1}{4}$ of $2rc$ the Surface of the Sphere. That is, the Surface of the Sphere is Quadruple of the Area of a great Circle of it.

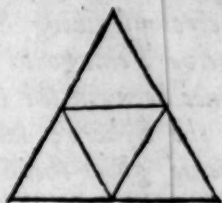
V. Wherefore, to $2rc$, the Convex Surface of the Cylinder, add rc equal to the Area of both its Bases (each of which is $\frac{rc}{2}$) you will have $3rc$; which shews you that the Surface of the Cylinder (including its Bases) being $3rc$, is to the Surface of the Sphere, which is, $2rc$, as three is to two: Or that the Sphere is $\frac{2}{3}$ of the circumscribing Cylinder, in Area, as well as Solidity.

34. Of all Solid Figures that can be encompassed or determinated by the same Surface, the greatest is a Spherical One, by *Art.* 13. of this Book, and *Art.* the last of Book the 4th.

35. That is call'd a *Regular Body*, whose Surface is compos'd of Regular and Equal Figures. And whose Solid Angles are all equal, as are——

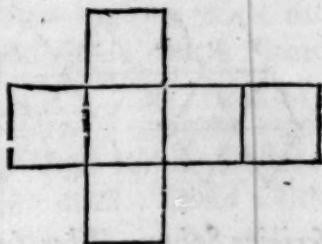
36. The *Tetrahedron*, which is a Pyramid, comprehended under four equal and equilateral Triangles; so that its Base is equal to each Side.

Wherefore its Solidity will be found by multiplying the Base by $\frac{1}{3}$ of the Altitude; which is the general way for all Pyramids.



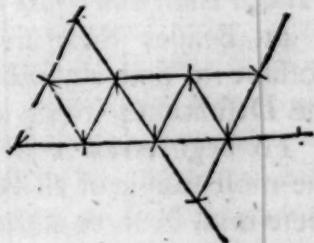
37. The *Hexahedron* or Cube, whose Surface is compos'd of six equal Squares, like Dice which are us'd in play.

Its Solidity will be found by Cor. of Art. 12.

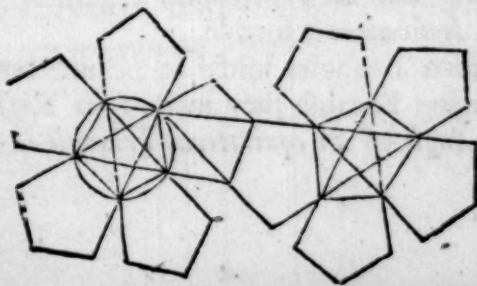


38. The *Octahedron*, which is bounded by eight equal and equilateral Triangles.

This Figure is two Pyramids put together at their Bases: Wherefore its Solidity is had by multiplying the Quadrangular Base of either (here they are both joined together in the middle of the Figure) by one third of the perpendicular Altitude of one of the joined Pyramids, and then doubling the Product.

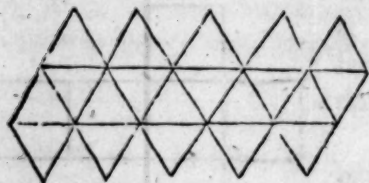


39. The *Dodecahedron*, which is contained under twelve equal and equilateral Pentagons.



This Figure consists of twelve Pyramids, with Pentagonal Bases, whose common Vertex is the Center of a circumscribing Sphere: Wherefore any one of these twelve Pentagonal Bases multiply'd by $\frac{1}{5}$ of the Distance between the Center of that Base, and the Center of the Sphere; and then that Product multiplied by twelve, gives the Solid Content of this Regular Body.

40. The *Icosihedron*, consisting of twenty equal and equilateral Triangles.



This Figure is composed of 20 Triangular Pyramids, all equal to one another, and whose Vertex is the Center of a circum-

scribing Sphere: Wherefore any one of the 20 Triangular Faces, multiplied by $\frac{1}{5}$ of the Distance between the Center of the Face and the Center of the Sphere, and that Product multiplied again by 20, gives its Solid Content.

41. Besides these five Regular Bodies, 'tis not possible to find any others that shall correspond to the Definition; which is thus demonstrated.

To begin with equilateral Triangles, which are the most simple of all Rectilineal Figures. Of these there must be three at the least to make a Solid Angle, and three of them joined together will just make the *Tetrahedron*. For those three Triangles meeting in a Point do form a Triangular Base similar and equal to the Sides; as appears by the bare Composition of the Figure. Four Triangles join'd together in a Point make the Angle of the *Octahedron*.

By joining five such Triangles together, the Angle of the *Icosihedron* is form'd.

But six such Triangles join'd in a Point can't make a Solid Angle: Because they make four Right Ones (for every Angle of an equilateral Triangle is $\frac{1}{3}$ of two

or

or $\frac{2}{3}$ of one Right Angle, either of which Fractions multiplied by six, gives four Right Angles.) Whereas every Solid Angle is made up of such plane Angles as all together must be less than four Right ones (5. 25.) So that with Triangles 'tis impossible to form any more Regular Bodies than these three.

Next, if you take Squares and join three of them together, they will make the Angle of the Cube: And there can no other Regular Body, but a Cube, be made with Squares, for four Squares join'd together, will not make a Solid Angle, but a Plane. (5. 25.)

If you join the Angles of three Pentagons together, you will constitute the Angle of the Dodecahedron: But four such Angles cannot make a Solid One.

And lastly, Three Hexagons joined together do make just four Right Angles, and therefore they cannot make a Solid Angle: And as for three Heptagons, or other Figures of yet more Sides, they can much less do it; (because their Angles being very Obtuse, three of them will exceed four Right Ones.) So that upon the whole 'tis plain, that of these five Regular Bodies, three are made of Triangles, one of Squares, and one of Pentagons, and there can be no other.



ELEMENTS

OF

GEOMETRY.

BOOK VI.

Of Proportion.



1. WHEN we speak of *Magnitude*, and say, that any Quantity is great, we always make a Comparison between that Quantity and some other of the same Nature, in respect to which we say that it is

Great.

Thus we say of an *Hill*, that 'tis Little; or of a *Diamond*, that 'tis Large; because we compare that Hill with others that are Higher, and in respect of them 'tis Little; and we compare that Diamond with others that are Little, and in respect of them, we say 'tis a Large one.

2. When we consider one Quantity in respect of another, to see what Magnitude it hath in comparison

son of that other: The Magnitude so found, is call'd its *Ratio* or *Reason*; tho' it would be more intelligible if it were call'd Comparison.

3. That Quantity which is compar'd with another is call'd the Antecedent; and *that other* with which it is compar'd, is call'd the Consequent.

4. When we consider four Quantities, and compare them (by Pairs) two with two; as *a* 4 with *b* 2, and *c* 6 with *d* 3. If we find that *a* hath as much Magnitude (or is as big) in comparison of *b*, as *c* hath in comparison of *d*; (i. e. *When we find that a is contained in, or doth contain b :: as often as c is contained in, or doth contain d*): Then we say, that their Ratio's are equal; that is, the Ratio that *a* hath to *b*, is equal to the Ratio of *c* to *d*: For as *a* is twice as big as *b*; so *c* is twice as big as *d*.



5. But if *a* 4 hath more Magnitude in respect of *b*, than *c* 6 hath in respect of *e* 5. That is, if as *a* 4 is twice as big as *b* 2, *c* 6 be found not to be twice as big as *e* 5: Then the Ratio's are unequal: And we say, *a* hath a greater Ratio to *b*, than *c* hath to *e*. So that to have a greater Ratio, is nothing but to have more Magnitude, or to be bigger, in respect of a second Term, than a third is in respect of a fourth.

6. The Equality of Ratio's is called Proportion; and when we find that of four Quantities or Numbers, the first hath as much Magnitude (*or is as big*) in respect of the second; as the third is in respect of the fourth; then we say, that those four Quantities are *Proportionals*.

The better to make the Mysteries of Proportion comprehended, which pass for the most difficult things in Geometry, as unquestionably they are most important, I will explain them by an Example; which

(in my Opinion) will render all those things very intelligible, which otherwise appear very perplex'd.

7. Let us imagine the Circle bAd to be describ'd by the Motion of the Line ob , round the Center o :



And at the same time, let the Circle cae be described by the Motion of a Point c , in the Line ob :

Let us suppose also that the Line ob be moved once round again, and at last to stand in the Position od .

Let the Ark dBb be called B, and the Ark eDc , be called D. Let A

be put for the whole outer Circle, and a for the whole inner one.

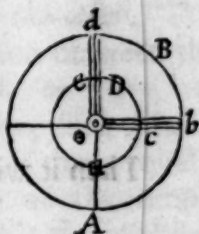
Now if we compare the whole Circle A with its Ark B, and the whole other Circle a with its Ark D, we shall find plainly, that the Circle A is just as big in respect of the Ark B, as the inner one a is in respect of the Ark D; and therefore if B be a fourth, or any other Part of the Circle A, D also will be a fourth, or the same proportional Part of its Circle a . Which we usually express by saying, as A is to B, so is a to D. And write it thus, $A:B :: a:D$, or $24:6 :: 8:2$.

8. If you should change the Order of the Terms, and compare B with A, and D with a ; you will find plainly that $B:A :: D:a$. So that supposing $A:B :: a:D$, we cannot but presently conclude by inverse Proportion, that $B:A :: D:a$.

9. If you change them so as to compare Antecedent with Antecedent, and Consequent with Consequent, you will find Alternately, that $A:a :: B:D$. And this is very plain; for if the whole Circle A be double, triple of, or in any other Proportion, to the Circle a , the Ark B must be also double, triple of, or in the same Proportion to the Ark D; for Aliquot Parts will be as their Wholes. This I say is plain,

plain, because the two Circles A and *a* are describ'd by the Motion of the Line *ocb*; so that while *b* describes the Circle A, *c* describes the inner Circle *a*; and while *b* describes the Ark B; *c* also describes the Ark D. And this by one common circular Motion; only the Point *c* moving much slower than the Point *b*, describes a Circle much less, in proportion to the slowness of its Motion: Thus also when the Point *b* shall have described the Ark B, the Point *c* in like manner will have describ'd the Ark D, which will be much less than B; in Proportion to the slowness of its Motion; in Numbers 24: 8::6:2.

10. If we compare the Differences between the Antecedents and Consequents with their Consequents; as for Instance, A less B with B, and *a* less D with D, we shall find they also are proportional: And that A less B: B:: *a* less D: D: 18: 6:: 6: 2.



For 'tis manifest that the Ark *bAd* (which is A less B) is to B as the Ark *cae* (which is *a* less D) is to D. And this is call'd *Proportion by Division*.

11. If we add the Antecedents and Consequents together; we shall find that A more B, is to B:: as *a* more D is to D. Which is call'd *Composition*. In Numbers 30: 6:: 10: 2.

12. And if we would say, that A A less B: *a*:: *a*: less D. This kind of Proportion's call'd *Conversion*. You may also infer by way of *mixing* the Terms, as some call it, That $A + B : A - B :: a + D : a - D$, or that $A + B : a + D :: A - B : a - D$, &c. That is 30: 18:: 10: 6 and 30: 10:: 18: 6, &c.

And it will be very convenient for the Learner to enure himself to all the Changes and Varieties of Proportion, and to have them ready in his Mind:

Because a great many Propositions in Geometry, as they have been delivered by the Ancients, and pursued by the Moderns that have trod in their Steps, are demonstrated by Composition, Division, Alternation, and intermixing of Proportion.

13. If ever so many Quantities are thus proportional: It will be as any one Antecedent to its Consequent:: So is the Sum of all the Antecedents to the Sum of all the Consequents, *v. gr.*

If $4:12::2:6$, $::3:9::5:15$: then shall $14:42::4:12$.

14.

If $a:b::c:d$.

$4:12::3:9$, and also,

$b:f::d:g$.

$12:36::9:27$.

Then it will be by Proportion of Equality,

$a:f::c:g$.

$(4:36::3:27)$.

The Reason of which is plain, if you consider, That since $b:f::d:g$: f and g must needs be either *similar aliquot Parts*, or *Equimultiples* of b and d . And therefore since a and c are to b and d , in the same Ratio as b and d are to f and g , a must also be in the same Ratio to f , the *Part* or *Multiple* of b :: as c is to g , the *Part* or *Multiple* of d .

If $a:b::c:d$.

$24:4::9:3$, and then,

$b:f::b:c$.

$4:2::18:9$.

Then

Then will $a : f :: b d$.

$$12 : 2 :: 18 : 3.$$

Which is called *Proportio ex aequo perturbata*; and this must be true: Because 12 containing 4 as oft as 9 contains 3, and 4 containing 2, as oft as 18 contains 9; 12 must contain 2 as often as 18 contains 3. Wherefore this is only the orderly Proportion of Equality disturbed, and therefore is by some called *Inordinate Proportion*.

15. If B be taken as often as D, *ex. gr.* 3 B and 3 D, we may conclude that $B : D :: 3 B : 3 D$, or as 10 B to 10 D, also $12 \frac{1}{2} B$, to $12 \frac{1}{2} D$. And so on in whatever Proportion the two Magnitudes B and D are multiplied, so they are multiplied equally, or that you take one as often as you take the other. For then there will be the same Proportion between the Magnitudes thus equally multiplied, as there was between the simple Magnitudes, before such Multiplication. And these Magnitudes, thus equally multiplied, are call'd *Equimultiples* of the simple Magnitudes B and D; and we say that *Equimultiples* are in the same Proportion as such simple Magnitudes, out of which they are compounded.

16. If B be divided in the same Manner as D is; and *ex. gr.* you take a fourth Part of B, and the like of D, or the tenth, or any other Part of B, and the same of D. Then will these Parts be proportional to their Wholes, $B : D :: \frac{1}{4} B$ (or $\frac{1}{10} B$) is to $\frac{1}{4} D$, or $\frac{1}{10} D$. All which is self-evident.

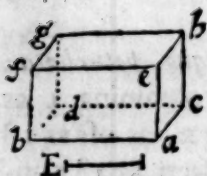
17. To multiply one Line by another is to make a Rectangled Parallelogram, whose two contiguous Sides shall be the two Lines given. Thus, if you multiply the Line A by B, 'tis the same thing as to make the Rectangle $a b c d$; whose Side $a b$ is equal to A, and $a c$ to b.



G 3

18. To

18. To multiply a Rectangle, or any other Surface by a Right Line, is to make a Rectangled Parallelopiped (or Prism) (5. 9.) whose Base shall be the Surface given, and its perpendicular Height the Line given.



Thus, to multiply the Surface $abdc$ by the Line E , is the same thing as to make a Solid $abfghc$, whose Base is the Surface given ad , and its Height ae or bf , equal to E , the Line given.

19. All Magnitudes may be express'd by Lines: As if one Magnitude be double or triple of another; or in any other *Ratio*, two Lines may easily be taken, of which one shall be double or triple of the other, or in any other like Proportion with those Magnitudes: So for Instance, to express two times, as one Hour and two Hours; or two Velocities, of which one shall be double to the other; you need only take two Lines, as a double of b ; and then you may say that a represents two Hours or Velocities, and b answers to one of each; and then you may proceed to compute with those two Lines, as with the Hours and Velocities themselves, &c.

20. To know the Proportion of Rectangles, the Ratio of the Length of one, to the Length of the other, and moreover, the *Ratio* of the Breadth of one, to the Breadth of the other must be known.

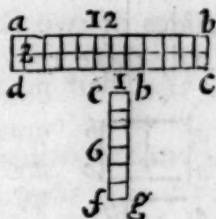
For Example; To know what Proportion the Rectangle ac hath to eg : 'Tis not



enough only to know that the Length ab is triple of eh ; but it must be known also, that ad is double of ef . For if ai be taken equal to ef , the Rectangle bi will be triple of eg , because ab is triple of eh , and ai equal to ef . And moreover, because id is also equal to ai , or ef (for

ad is supposed to be double of ai , and of ef) the Rectangle ic shall also be triple of eg ; so that the whole Rectangle ac is twice triple of the Rectangle eg ; that is, sextuple of it, or containing it six times. And what we say now only of the double or triple *Ratio* of their Breadths and Lengths, is also to be understood of any other *Ratio*, be it what it will: For if ab be quadruple of eb , and ad triple of ef , the Rectangle ac , will be three times quadruple of the Rectangle eg ; that is duodecuple of it, or doth contain it twelve times.

But if ab be duodecuple of eb , and at the same time ef be triple of ad , then there is a certain Compensation made: For if Respect were had to their Breadths ab and eb only, the Rectangle ac would exceed the other, nay indeed contain it 12 times: Nevertheless this Excess is lost (*in some Measure*) in respect of their Altitudes or Heights ad and ef , which if only consider'd, the Rectangle eg would be triple of ac . But then when we come to compare these several Excesses and Deficiencies together; we shall find that the Rectangle ac being one way 12 times greater, and the other way three times less than eg , will be at last but only four times as great.



21. And this is what we mean, when we say, that all Rectangles are to each other in a *Ratio compounded* of that of their Sides; for if ab be triple of eb , and ad double of ef , the Rectangle ac shall be to the Rectangle eg in a *Ratio compounded* of the triple and the double, that is, it shall be twice double, or twice triple, or, in one Word, sextuple. So also if ab were quadruple of eb , and ad

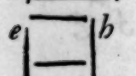
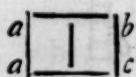


triple of ef ; the Rectangle ac would then be to eg in a *Ratio* compounded of the quadruple and the triple; so that it would have been three times quadruple, or, four times triple, or in one Word, duodeuple of eg .

Moreover, if ab were duodeuple of eh , and ad subtriple of eh , (that is, if ef be triple of ad) the *Ratio* of the Rectangle ae to eg would be compounded of the duodeuple and subtriple *Ratio*; so that ac would have been 12 times subtriple of, or, in one Word, quadruple of eg .

If you take the third Part of a Crown 12 times, it will make, or be equal to four whole Crowns: So that four Crowns are 12 times subtriple of one Crown; that is, do make 12 Thirds of a Crown.

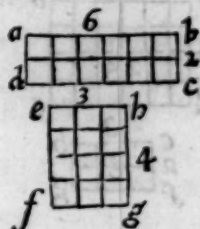
22. From whence it will appear, that if the Sides of two Rectangles are reciprocally proportional, those two Rectangles are equal: For if ab be



double to eh , and reciprocally hg be double to cb : Or if ab be triple of eh , and then hg be triple of bc ; or, in a Word, if whatever *Ratio* ab hath to eh , hg hath back again the same *Ratio* to bc ; 'tis plain, that as much as the first Rectangle ac exceeds the other in

Length, just so much is it exceeded by the other in Breadth; so that the Length of one compensates for the Breadth of the other, and consequently they must be equal. And from hence is deduced this most useful and important Proposition: That,

23. If four Quantities (*or Numbers*) be proportional, the Product arising from the Multiplication of the two middle Terms, is always equal to that which is made by the Multiplication of the two Extrems. As if $ab:eh::hg:bc$.



I say, from the Multiplication of the Extrems ab by bc there is produced the Rectangle ac :

And by multiplying the middle Terms eh and hg , there is produced the Rectangle eg ; and those two Rectangles ac and eg are equal. (6. 22.) *Because, as much longer as ab is than eh , just so much longer is hg than bc .* On which is founded the Reason of the Golden Rule.

COROLLARY.

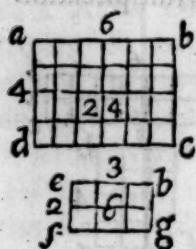
Hence, if in two Ranks of *Discreet* Proportionals, the four middle Terms are the same: As if $a:b::c:d$, and then also $e:b::c:f$. I say, it will be as $a:e::f:d$. So will reciprocally f , be to d : For since the middle Terms are the same in both, the Rectangle ad will be equal to ef , and consequently their Sides must be reciprocally proportional; that is, $a:e::f:d$.

What is thus done by Lines and Rectangles, may be done by any Quantity whatsoever; because all Quantities can be expres'd by Lines, and all Multiplications of Magnitudes by Multiplications of Lines, *i. e.* by Rectangles. (6. 24.)

24. When Rectangles have their Sides directly proportional, so that $ab:eh::ad:ef$, then is the Rectangle ac to the Rectangle eg , in a *Duplicate Ratio*, to that of their Sides: For the *Ratio* of ac

to

to eg , is compounded of the *Ratio* of ab to eh , and of



the *Ratio* of ad to ef (6. 26.) But the *Ratio* of ab to eh is in this Case (by the Supposition) the same as the *Ratio* of ad to ef ; so that to gain the *Ratio* which the Rectangle ac hath to eg , we need only take twice the *Ratio* of ab to eh . For Example, if as here ab be double to eh ,

and ad double to ef , the Rectangle ac shall be twice double, that is, quadruple of the Rectangle eg . And if ab had been of eh , and consequently ad triple of ef : Then the Rectangle ac would have been three times triple, that is nine times as big as eg : Or if ab had been quadruple of eh , ac would have been 16 times as great as eg .

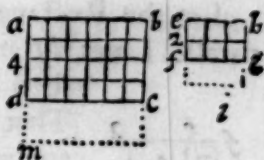
25. If a third Line be taken as no ; and it be so proportional that $ab:eh::eh:no$. Then $n|.....|o$ shall the two Rectangles ac and eg be to one another, as the two Lines ab and no : (vid. Fig. *Preced.*)

For ab is to no in a duplicate *Ratio* of ab to eh . And if ab had been (as it is double) triple or quadruple of eh : Then would ab have been in a *Ratio* three times triple; or four times quadruple of (that is, 9 or 16 times as great as) the third Proportional no .

26. Those Rectangles which have their Sides thus proportional: That $ab:eh::ad:ef$, are called *Similar*, whose *Homologous Sides* are those which answer each to other in the Proportion, as ab and eh , or ad and ef : For as ab is the greatest Side of the Rectangle ac , so eh is also the greatest Side of the Rectangle eg .

27. All Squares are similar Rectangles. For 'tis plain that if ab be double or triple of eb , am must also be double or triple of hi :

Because am is equal to ab , and hi to eb .



28. All similar Rectangles are to each other as the Squares of their Homologous Sides. I say the Rectangle ac is to the Rectangle eg :: as the Square bm to the Square ei . For as well Squares as Rectangles are to one another in a duplicate *Ratio* of ab to eb (6. 29. 30.)

29. To know the *Ratio* between two solid Rectangles or *Parallelopipeds*, there ought to be known the several *Ratio*'s that their Bases and Heights have to each other; because the *Ratio* of one Solid to another is compounded of the *Ratio*'s of their Lengths and Breadths, and Thicknesses or Heights; as is easy to conceive, if that be well understood which hath been said about the Proportions of Rectangles. For if one *Parallelopiped* hath its Base double to the Base of another, and its Height triple of the Height of the other; the former will be twice triple, or three times double, or, in one Word, sextuple of the latter.

30. If the *Bases* of two *Parallelopipeds* be *Reciprocally* as their *Heights*, those *Parallelopipeds* are equal: Which is proved by the 27th of this Book; for as much as one exceeds the other in Breadth and Length, so much doth the other exceed it in Height.

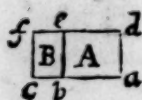
31. When *Parallelopipeds* have all their Sides proportional, they are called *Similar*; and they are in a *Triplicate Ratio* of their Sides, as it hath been proved of Rectangles, that they are in a *Duplicate Ratio* of their Sides.

32. Similar *Parallelopipeds* are to one another as the Cubes of their Homologous Sides; for both Cubes and *Parallelopipeds* are in a *Triplicate Ratio* of their Homologous Sides.

33. All

33. All Rectangles, having the same or equal Heights, are to one another as their Bases, and having the same Bases their Heights are equal.

Let the Rectangles A and B be between the same parallel Lines df and ca ; so that ad be equal to cf :



then do I say, that $A : B :: ab : bc$:

That the Rectangle A is to the Rectangle B, as the Base ab to the Base bc :

And that if, for Instance, ab be double to bc , then shall A be double to B. For

A is nothing but the Line ba multiply'd by da . (6. 17.) and B is nothing but the Line cb multiply'd by the same Line ad , or (which is all one) be or fc . Wherefore (6. 15.) $A : B :: ab : bc$.

34. All *Parallelograms* which are between the same Parallels (or which have the same Height) are as their Bases. I say, the Parallelogram $bg ::$ as the Base ab is to the Base bc .



For having made the two prick'd Rectangles on the same Bases, those will be equal to the Parallelograms (by 3. 14.) But

those Rectangles are as their Bases (by the Precedent.) Wherefore the Parallelograms must also be as their Bases: That is, $eb : bg :: ab : bc$.

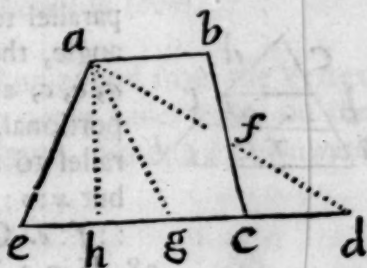
35. All Triangles (which have the same Heights) or are between the same Parallels, are as their Bases; for they are Halves of Parallelograms (3. 8.)

36. When Triangles (as those in the following Figure) have their Bases on one and the same Line, and their Vertices or Tops meeting in the same Point; they are taken to be between the same Parallels, as ade and cde , and ade and bde (because they have the same perpendicular Height.)

PROBLEM I.

Hence may a Trapezium, as $abce$, whose two Sides ab and ec are parallel, be divided into any given *Ratio*.

For take $cd = ab$ and draw ad , then will the Triangles abf and fcd be equal (by 14. 2.) and consequently the



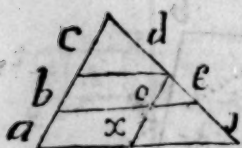
Triangle $ead = \text{Trapez. } eabc$. Wherefore if you divide ed the Base of the Triangle ead into any Number of Parts, or according to any *Ratio*, Lines, drawn from the Vertex to such Divisions of the Base, will divide the Triangle ead , and consequently the Trapezium, in the same *Ratio*.

37. If in any Triangle a Line be drawn parallel to the Base, that Line shall cut the Legs proportionally. Let the Triangle be abe , and let the Line de be parallel to bc .



I say, that $ad : ae :: ab : ac :: db : ec$, &c. Draw the Lines dc and eb , then shall the Triangle ecd be to ead , as the Base ec is to ae . (6. 40. 46.) So also the Triangle deb is to $ead ::$ as the Base db is to da . But the Triangle ecd is equal to deb (3. 15.); wherefore the Triangle bde (or ced) is to the Triangle $ead ::$ as $bd : da ::$ or as ce to ca . Therefore also must $bd : da :: ce : ea$, because both the *Ratio* of $bd : da$, and also that of $ec : ea$, are the very same with that of the Triangle bed or ced , to the Triangle ade ,

COROLLARY.



If many Lines are drawn parallel to the Base of any Triangle, the Segments of the Sides a, b, c , and d, e, f , will be proportional, for drawing ox parallel to $abc: b = o$ and $a = x$, but $x: o :: f: e$: Wherefore $a: b :: f: e$. Q. E. D.



38. If in a Triangle, as acb , you draw a Line de parallel to the Base cb , I say, that $ed: cb :: ae: ac ::$ or as $ad: ab$. For drawing ef parallel to ab ; fb will be equal to ed . (3. 9.) But by the Precedent $fb: cb :: ae: ac$. Wherefore $ed: (fb) eb :: ae: ac$, or as ad to ab .

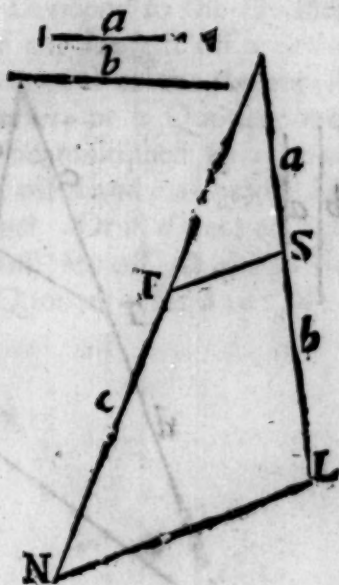


PRO-

PROBLEM I.

Two Lines a and b being given, to find c , a third Proportional to them.

Make any Rectilineal Angle, and from the Vertex or Top of it, set the two given Lines down on the Legs, as you see in the Figure. Set also b downward



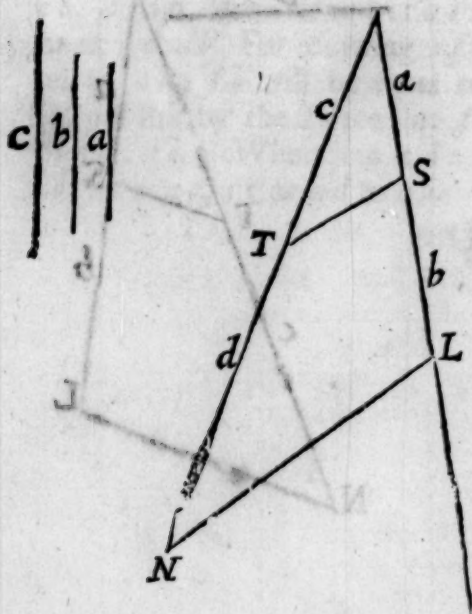
from S to L , join ST , and draw NL parallel to it; so shall TN be c , the Line sought; For $a : b :: b : c$, by this Proposition.

PRO-

PROBLEM II.

If three Lines, as a, b, and c, had been given, to find a fourth Proportional (as d) to them, or to work the Rule of Three in Lines, you must proceed thus :

Set the two first Lines *a* and *b* from the Vertex down on the same Leg; and then set *c* the third Line, from the Vertex on the other Leg : Draw the



Line *TS*, and through the Point *L* draw *LN* parallel to it; so shall *TN* be equal to *d*, the fourth Proportional sought; for $a:b::c:d$, by the precedent Propositions.

P R O.

P R O B L E M III.

And this way 'tis very easy to find a Line that shall express the Product of any two Numbers or Quantities: Or the Quotient of one divided by the other.

For since in all Multiplication, as 1 is to the Multiplier :: So is the Multiplicand to the Product: And since in Division, as the Divisor is to 1 :: So is the Dividend to the Quotient: You may take your 1 of any Length off a Scale, and finding a fourth Proportional to the three first Terms, that shall be a *Product*, or a *Quotient* required: Thus if b were to be multiplied by c , make a equal to Unity, and set off b and c as before shew'd, so shall d be the *Product*. Or if d were to be divided by b , take $a = 1$, and set off all things as before; so shall c be the *Quotient*; for $b : a :: d : c$.

PROBLEM IV.

To divide a given Line ab into any Number of equal Parts: As suppose into Six.

Make at a and b any two equal Angles, and on the Legs da and cb run with a Pair of Compasses five equal Divisions (for they must be always one less in Number than the required Division or Parts of the Line given) drawing also Lines across from one Point to the other, as you see in the Figure; so shall those Lines divide the given Line ab into the six Parts required: For the crossing Lines being parallel one to another, must divide ab in the same Proportion as ad and bc are divided.



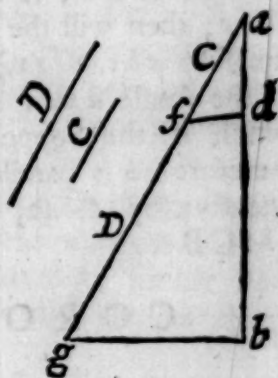
PROBLEM V.

To divide a given Line ab into two Parts, so that they shall be to each other as the Line C to D ; or in any given Ratio.

Make any Angle with the given Line ab , and set the Line c from its Vertex a to f . And set the Line D from f to g ; draw the Line gb , and thro' f , a Parallel to it, as fd : So shall the Point d divide ab in the Ratio required: For $C : C :: ad : db$.

And much the same way may you cut off from any given Line ab any Part or Parts required; as suppose $\frac{2}{7}$.

Make any Angle as gab as before, and set on the Leg ag , ag equal to five Parts taken off from any Scale: Then set two such Parts from a to f , join gb , and draw fd parallel to it; so shall ad be equal to $\frac{2}{7}$ of ab .



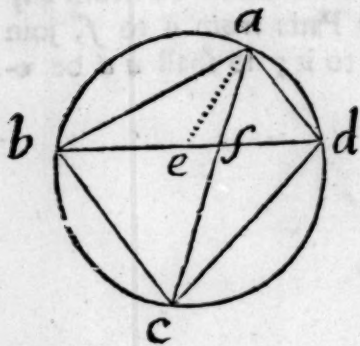


38. Those Triangles are called *Like* or *Similar*, which have all their three Angles respectively equal to one another, or which are *Equiangular*: v. gr. If the Angle A be equal to a , the Angle B to b , and C to c , then the whole Triangle ABC is *Like* or *Similar* to the Triangle abc .

39. All similar Triangles have their Sides about the equal Angles proportional. I say, $AB : ab :: AC : ac :: BC : bc$, &c. For take in the greater Triangle ABC, Ab equal to ab , and Ac equal to ac ; then will the Triangle Abc be every way equal to abc (2. 11.) and the Angle Abc is equal to the Angle abc ; wherefore it will be also to B, which by the Supposition was equal to abc , and therefore cb is parallel to CB (1. 31.) and consequently (by 6. 42, 43.) $Ab : AB :: Ac : AC :: cb : CB$.

COROLLARY I.

If a Quadrilateral Figure, as $abcd$, be inscribed in a Circle, the Rect-angle under the Diagonals bd and ac , is equal to both the Rectangles under the opposite Sides:



That is, $ac \times bd = ba \times cd + ad \times bc$, make the Angle $bae = ead$; and then adding the Angle eaf to both, the Angle $baf = ead$: And the $\triangle aad$ similar

to $\triangle baf$: Then will $ac : cd :: ba : be$. (by this Prop.) Wherefore $ac \times be = cd \times ba$. Again al-

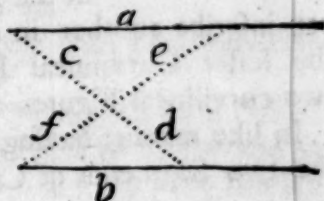
fo

so $ad:de::ac:cb$. Wherefore $ad \times cb = de \times ac$. But $ac \times be + ac \times ed = ac \times bd$. Wherefore $ac \times bd = ba \times dc + da \times bc$. Q. E. D.

COROL. II.

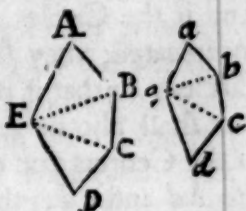
The Segments of Lines intersecting each other between two Parallels, are proportional:

That is, $c:d::e:f$, for by similar Triangles $c:e::d:f$; wherefore alternately $c:d::e:f$. Wherefore $cf = de$.

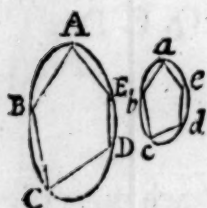


40. All similar Triangles are in a Duplicate Ratio of, or as the Squares of their Homologous Sides; for similar Triangles are the Halves of similar Parallelograms; wherefore they must be as their Wholes.

41. Similar Polygons are those which having an equal Number of Sides, have all the several Angles in one, equal to those in the other, and also the Sides about those equal Angles proportional. As if the Angle A be equal to a , B to b ; and moreover $AB:ab::BC:bc::CD:cd$; then those two Polygons are similar.



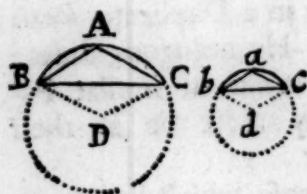
42. And among *curvilinear* or *mixt Figures*, those are *similar* in which you may inscribe, or about which



you may circumscribe similar Polygons; so that any Polygon being inscribed or circumscribed about one Figure, you may inscribe or circumscribe a similar one about the other. For instance, if having inscribed any Polygon, as ABCDE,

in the greater *curvilinear* Figure you can inscribe another in all respects similar to it in the lesser Curvilinear Figure *abcde*, then those two curvilinear Figures are similar.

In like manner having taken two mixt Figures, as the two Segments of Circles BAC and bac;



having inscribed in one any Triangle at Pleasure, as BAC; if then you can inscribe in the other Segment another Triangle *bac*, that shall be similar to the former; then shall those two Segments be similar Figures.

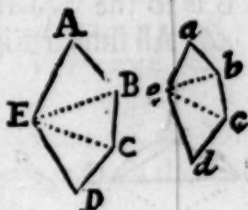
And if the Circles of which they are Segments be compleated, they shall be similar Parts of those two Circles; so that if BAC be a third Part of its Circle, *bac* shall also be a third Part of its Circle: And if to the Centers you draw the Lines BD and CD, and also *bd* and *cd*; the Angles D and *d* shall be equal. (See 4. 11. and the following Propositions.)

COROLLARY.

The Peripheries of Circles are as their Diameters: For as $AB : ba :: BD : bd :: 2 BD : 2 bd ::$ so it will be of every side of the inscribed or circumscribed Polygon; wherefore the Sum of them all, that is the Peripheries, must be in the same Ratio.

43. All Circles are similar Figures.

44. All similar Polygons may be divided into an equal Number of similar Triangles. Let the similar Polygons be $ABCDE$ and $abcde$; and let the first be divided into Triangles by the Lines EB and EC (3. 14.) I say that if the other be also divided into Triangles by the Lines eb and ec , all the Triangles in one shall be (*respectively*) similar to those in the other.



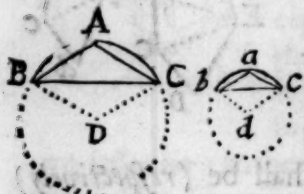
For instance, I say the Triangle abe is similar to ABE : for the Angle a is equal to A (by the Supposition) and also $AB : ab :: AE : ae$ (by the same) wherefore the Triangle ABE is similar to abe . (6. 46.) Again, the Angle EBC may be proved equal to ebc ; because the Angle ABC is (by the Supposition) equal to abc , and it was proved (*in the last Step, where the Triangle ABE was proved similar to abe*) that the Angle abe is equal to ABE ; wherefore from equal things taking away equal, the Angle EBC remains equal to the Angle ebc . In like manner the Angle ecb is prov'd equal to ECB , and consequently (6. 45.) the whole Triangle ebc will be similar to EBC ; and so of the rest.

Hence the Practice of making on a Line given a Polygon similar to one assigned is derived. For dividing the given Polygon into Triangles, make a Figure, consisting of a like Number of similar Triangles, on the given Line.

45. All similar Polygons are to one another in a Duplicate Ratio of, or as the Squares of their Homologous Sides. I say, as the Square of AB is to the Square of $ab ::$ So is the whole Polygon $ABCDE$ to the Polygon $abcde$. For since all the Triangles in one Polygon are similar to those

in the other (6. 51.) all in one Polygon will be to all those in the other in a duplicate *Ratio* of any of their Homologous Sides; that is, as the Square of AB is to the Square of ab .

46. All similar Figures, even Curvilineal ones, are to one another as the Squares



of any Side of any similar Figures, which can be inscribed or circumscribed about them, *v. gr.* Let there be two Circles, in which are inscribed two similar Triangles abc and ABC : I

say, the whole Circle ABC , is to the Circle abc : So is the Square of BC to the Square of bc , or, which is the same thing, as the Square of the Radius BD to the Square of the Radius bd . For in or about the Circle abc may be inscribed or circumscribed any Polygon you please (or at least such an one may be imagined) (4. 30.) But every Polygon inscribed in abc will have a less *Ratio* to the Circle ABC than the Square of bc hath to the Square of BC : and every one circumscribed about abc will have a greater *Ratio* to the Circle ABC , as is easy to prove by the Precedent, and from what hath been said of Circles in the fourth Book. Wherefore all similar Figures, &c.

COROLLARY I.

I. Circles are to each other as the Squares of their *Radii* or *Diameters*: for suppose a Circle whose Radius is r , and then another Circle greater, or less than that, and call its Radius R , then will its Diameter be $2R$: then whatever the *Ratio* of the Diameter ($2R$) be to the Periphery, let it be expressed by

by the Letter e , then will $2 R e$ (or e times the Diameter) be the Periphery; and half of this, *viz.* $R e$, multiplied by R will be the Area, *viz.* $R R e$. And by the same Method of reasoning, the Area of the other Circle will be $r r e$. But certainly $R R e : r r e :: R R : r r :: 4 R R : 4 r r$. Wherefore, &c.

II. Hence 'tis plain, that the Square of the Diameter of any Circle is to the Area of it, as the Diameter is to $\frac{1}{4}$ Part of the Periphery.

For $4 R R : R R e :: 2 R : \frac{2 R e}{4} (= \frac{1}{2} R e)$

As is plain by multiplying the Extreams and mean Terms by one another.

III. Hence also 'tis plain (again) that the Peripheries of Circles are as their Diameters.

That is, $2 R : 2 r :: 2 R e : 2 r e$.

IV. And since the Area of every Circle is $r r e$ (that is, the Product of the Square of the Radius multiplied into the Name of the Ratio, between its Diameter and Periphery.) A very ready way (for common use) to find the Area of a Circle whose Radius is given, will be to multiply the Square of the Radius into this or such like Decimal 3.1. Thus suppose the Radius 9 Inches: $81 \times 3.1 = 251.1$; which is very nearly the Area in square Inches, tho' something less.

47. All this may be apply'd to Solids. And therefore *similar Solids* are *such*, as have their Angles all equal, and the Sides about those Angles proportional; or (if they are of a spherical or of any spheroidal Figure) *such*, as can have similar Solids inscrib'd or circumscrib'd in or about them, &c.

48. Simi-

48. Similar Solids are to one another in a (*Tripliate Ratio* of, or) as the *Cubes* (of their *Homologous Sides*, &c.) See 6. 36, 37, &c.

(And therefore all Spheres must be to one another as the Cubes of their Diameters, &c.) Which may be easily thus proved; the Solidity of the Sphere may be expressed after this manner; by what is said in the Corollaries in p. 75, 76.

The Area of a great Circle of the Sphere, whose Radius is R or r , being $R R e$ or $r r e$ (by *Cor. 1. Art. 53.*) 4 times that will be the Surface of each Sphere; that is, $4 R R e$ the Surface of the greater, and $4 r r e$ the Surface of the lesser; and multiplying the Surface by $\frac{1}{3}$ of the Radius, the Solidities will be $\frac{4 R R R e}{3}$ and $\frac{4 r r r e}{3}$: Which two

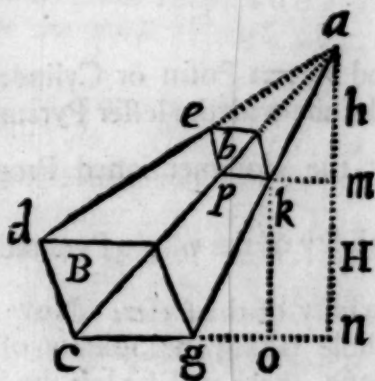
Quantities being multiplied and divided by the same, will be in the same *Ratio*, when without such Multiplication and Division: That is $\frac{4 R R R e}{3}$

$\frac{4 r r r e}{3} :: R R R : r r r$. That is, Spheres are as the Cubes of their Radii, and consequently, as the Cubes of their Diameters. Q. E. D.

P R O B L E M.

To find the Solidity of the Frustum of a Pyramid or Cone, cut by a Plane parallel to the Base, having given the two Bases together with the Height of the Frustum.

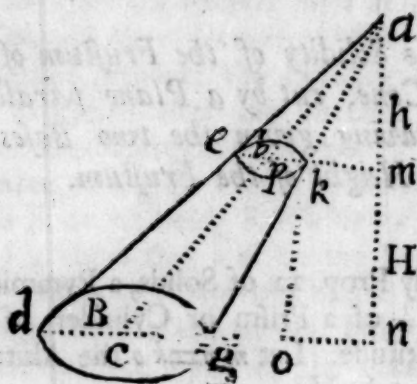
Solution. By Prop. 32. of Solids, a Pyramid or Cone is equal to $\frac{1}{3}$ of a Prism or Cylinder of the same Base and Altitude. Let $mn = ko$ the Altitude of the Frustrum, be called H , and ma the Height of the Top-piece wanting h ; the Greater Base of the Frustrum B , and the lesser b ; the Triangles $ap k$ and $ac g$ are similar (cg and pk being parallel *ex Hyp.*) wherefore $cg : ga :: pk : ka$, and alternately $cg : pk :: ga : ka$. But $ga : ka ::$



$na:ma$ (from the Similarity of the Triangles, gan and kam) wherefore *ex æquo* $cg:pk::na:ma$, and by Division $cg - pk:pk::na - ma:(=mn)ma$; which put into Symbols (putting cg the side of the Base $=S$, and $pk=s$) will stand thus: $S-s:s::H:\frac{S-H}{S-s}=h$. Wherefore having found

the Height of the little Pyramid or Cone which is wanting, I say, having found it in known Terms, it will be easy to find the Solidity of the Frustum; for multiplying the Base B into the whole Height $H + h$, the

the Product $BH + Bb =$ a Prism of the same Base and Altitude with the whole Pyramid or Cone;



and $bb =$ a Prism or Cylinder of the same Base and Altitude with the lesser Pyramid or Cone. Wherefore,

by the aforementioned Proposition, $\frac{BH + Bb}{3} =$

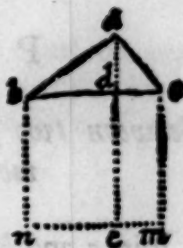
Solidity of the whole Pyramid or Cone, and $\frac{bb}{3} =$

Solidity of the lesser. Now from the Solidity of the whole taking the Solidity of the lesser Pyramid or Cone, there will be left the Solidity of the Frustum

required, viz. $\frac{BB + Bb - bb}{3} =$ Solidity of the Frustum.

The Theorem in Words is this: *Multiply the greater Base by the whole Height, and from the Product subtract the upper Base multiplied by the Height of the Top-piece wanting, and $\frac{1}{3}$ of the Remainder will give the Frustum.*

49. If in a Rectangle Triangle abc , a Line as ad be drawn from the Vertex or Top of the Right Angle, perpendicular to the Base, Hypothenufe, or longest Side bc , it shall divide the Triangle abc into two other Rect-angled ones, abd and dac , which will be similar to each other, and to the whole bac . For, 1. All the three Triangles have one Right Angle. 2. The Triangles abc and abd have the Angle b common to both: Wherefore they are similar (6. 45.) 3. The Triangles abc and adc have also the Angle c common to both: therefore they two are similar; and lastly, abd and adc being both similar to one third Triangle abc , will be so to each other.



50. The Perpendicular ad is a mean or middle Proportional between bd and dc . That is, $cd : da :: da : db$. For the Triangles cda and bda being similar (by the last) cd (the lesser Leg of the Triangle cda) shall be to ad (the greater Leg) :: As the same ad (the lesser Leg of the other Triangle adb) is to bd the greater Leg. (6. 46.)

51. The Square of ad is equal to the Rectangle made between cd and db . For, since $cd : da :: da : db$ (by the last) the Rectangle of the Extreams cd and db , is equal to the Rectangle of the mean Terms da and da (6. 28.) But the two sides of that Rectangle being equal, because 'tis only da taken twice, that Rectangle must be the Square of da ; and so it may be laid down as an universal Theorem, that, &c. the Square of the Perpendicular drawn from the Vertex of any Rectangle Triangle to the Hypothenufe, is equal to the Rectangle under the Segments of that Hypothenufe.

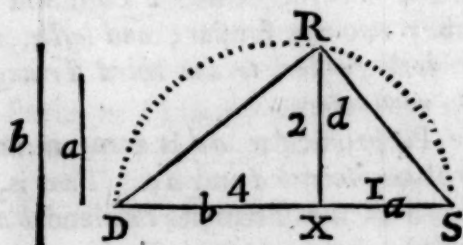
52. The

52. The Square of a mean Proportional is always equal to the Rectangle of the Extreams.

PROBLEM I.

Between two given Lines a and b , to find a mean Proportional, as d .

Join a and b both in one Line, which make the Diameter of a Circle; and then at the Point x , where the given Lines join, erect a Perpendicular as d ; that



shall be the mean Proportional required. For the Angle DRS being a Right One (as being in a Semicircle) $b : d :: d : a$, by Prop. 57.

PROBLEM II.

And thus may you find a Line equal to the Square Root of any Number or Quantity, by finding a mean Proportional between it and 1. For if $b = 4$, and $a = 1$; then will $d = 2$, equal to the Square Root of b .

PROB.

PROBLEM III.

Thus also may a Square be found equal to any Rectangle given, by finding a mean Proportional between its Sides, which shall be the Side of the Square required.

PROBLEM IV.

To find a Square equal to any Triangle.

Find a mean Proportional between a Perpendicular let fall from any Angle to an opposite Side, and the half of that Side; and that shall be the Side of the Square required.

53. *A Rectangle being given to make another Rectangle equal to it, which shall have a Length given.*

Let the Rectangle given be ac , and let

it be required to make another equal to it, the Length of one of whose sides shall

be the Line ef : Here are now three Lines given, viz. ab and bc (which are the sides of the Rectangle given) and

ef , which must be one side of the Rectangle required. Therefore a fourth Line must be found which shall be the other Side of the Rectangle sought: Which is done by finding a fourth Proportional to the three given Lines (6. 43.) which let be eh . So that $ef : ab :: bc : eh$; and then I say, the Rectangle fh is equal to db , and is the Rectangle required. (6. 27.)



N. B. *This is called Application of the Rectangle, equal to a Right Line ab , vid. Eucl. p. 6. e. 6.*

54. To express a Rectangle, you need use but three Letters, *v. gr.* When we say the Rectangle bdc , we mean a Rectangle, one of whose Sides is bd , and the other dc . But if we say the Rectangle bcd , we then mean a Rectangle, one of whose Sides is bc , and the other cd .

55. In every Rectangle Triangle the Square of the Hypothenufe is equal to the (Sum of the) Squares of the two other Sides (or to the Sum of the Squares of the Legs.)

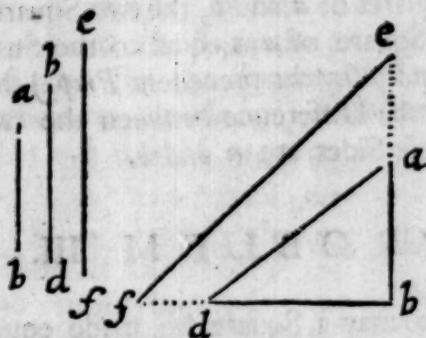


Let the Square bm be divided by the Perpendicular ade into the two Rectangles dm and dn . I say that the Rectangle dm is equal to the Square of ac , and the Rectangle dn , to the Square of ab : And that by consequence the whole Square bm is equal to the Sum of the Squares of ab and ac . For, 1. The two Triangles adc and abc being similar (6. 56.) $dc : ca$ (in the lesser Triangle dca) :: as the same $ac : cb$ in the greater Triangle acb . Wherefore ac is a mean Proportional between dc and cb (or cm) and consequently the Square of ac is equal to the Rectangle bcd , or dcm , that is dm .

And after the very same manner may ab be prov'd to be a mean Proportional between bd and bc (that is bn , &c.) (for the Triangle abd being similar to abc , db the lesser Side in one will be to ba the greater Side, as that ba (now the lesser Side in the other Triangle abc) is to bc the greater Side: That is, $db : ba :: ba : bc$ (or bn) and consequently the Square of ab is equal to the Rectangle dbn , or dn . And so both the Squares together, of ba and ac , or their Sum is equal to the Square of the Hypothenufe. Q. E. D.

PROBLEM I.

Hence any two, or more Squares, may easily be added together into one Sum. Let ab , bd , and ef , be the Sides of three given Squares; place ab and bd

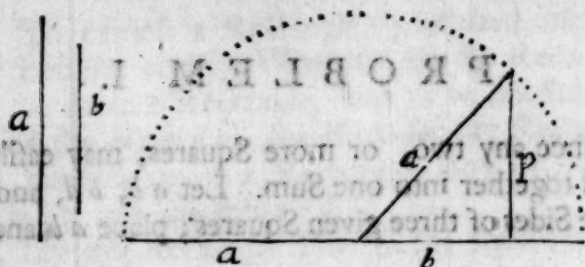


at Right Angles, and draw the Hypothenufe ad , whose Square will be equal to the Sum of the Square of db and ab . Then set da from b to e , and the given Line ef , from b to f : So shall the Hypothenufe fe , be the Side of a Square equal to the Sum of the three given Squares.

PROBLEM II.

Or if two Squares be given, you may subtract one from the other, and find a Square equal to the Difference between them.

Let a and b be the Sides of the given Squares; make (the longest Line) the Radius of a Circle, and set b from the Center on the same Right Line with a ; at the End of b , erect the Perpendicular p , which will be the Side of a Square equal to the Difference be-



tween the Squares of a and b , the two Squares given; for since the Square of a is equal to the Sum of the Square of b and p (by the precedent *Prop.*) the Square of p must be the Difference between the two given Squares, whose Sides are a and b .

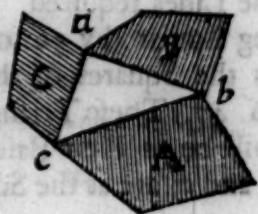
PROBLEM III.

Hence also may a Square be made equal to any given Polygon, or irregular Right-lined Figure: By reducing the Figure into Triangles; finding Squares equal to those Triangles; and then one Square at last equal to the Sum of all those Squares.

Or by making Rectangles equal to those Triangles, which shall have all the same Height; then joining those Rectangles together, so as to make one great one equal to them all; and lastly, make a Square equal to that Rectangle.

59. If upon the three Sides of a Rectangled Triangle are made three similar Figures, and those similarly posited, the greatest shall be equal to the other two.

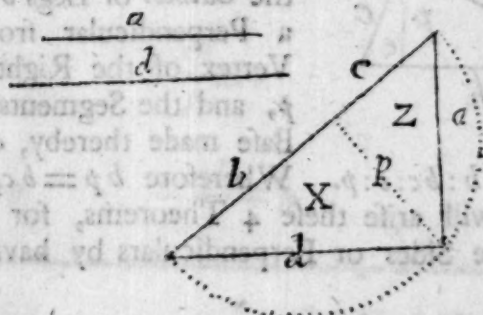
For the three Figures being similar, are as the Squares of their Homologous Sides (6.53.) And therefore the Figure A shall be to B and C, as the Square of bc is to the Squares of ab and ac . But the Square of bc is equal to those two Squares (by the last) therefore the Figure A is equal to both B and C together.



PROBLEM I.

To find two Lines, b and c , which shall have the same Ratio to one another, as two given Squares, similar Triangles, similar Polygons, or Circles.

Let a and d be the Sides of the two given Squares, Triangles, Polygons; or the Diameters or Radius's of the Circles given: Set them at Right Angles to

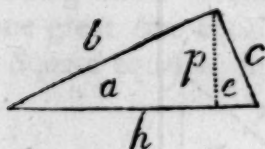


one another, as you see, and draw the Hypothenuſe $b + c$, to which let fall the Perpendicular p , which ſhall divide the Hypothenuſe into two Parts b and c , the Lines required. For the Triangles Z and X being ſimilar (by 56. of the 6.) will be to one another as the Squares of their homologous Sides a and d , (6. 47.) Theſe Triangles alſo having the ſame Height, will be as their Baſes (6. 42.) wherefore their Baſes b and c , are as the Squares of d and a . Q. E. D.

PROBLEM II.

This Problem may be inverted thus; To make two Squares, Triangles, &c. having the Ratio of two given Lines, b and c , or in any given Ratio.

Join the Lines into one continued Line, and then make that the Diameter of a Circle; from the Point where b and c join, erect a Perpendicular to the Curve as p , then draw d and a , and they ſhall be the Sides of the Squares, Triangles, ſimilar Polygons, or the Diameters of the Circles required.



In a Right-angled Triangle let the *Hypothenuſe* be b , the *Catheti* or Legs b and c , a Perpendicular from the Vertex of the Right Angle p , and the Segments of the Baſe made thereby, a and e .

Then 1. $b : b :: c : p$. Wherefore $bp = bc$, from whence will ariſe theſe 4 Theorems, for finding any of the Sides or Perpendiculars by having the reſt.

$$\left. \begin{array}{l} 1. b = \frac{bc}{bp} \\ 2. b = \frac{bp}{c} \end{array} \right\}$$

$$\left. \begin{array}{l} 3. c = \frac{bp}{b} \\ 4. p = \frac{bc}{b} \end{array} \right\}$$

$$\left. \begin{array}{l} 2. a. b :: b. b \\ e. c :: c. b \end{array} \right\} \text{Wherefore } \left\{ \begin{array}{l} bb = ab \\ cc = eb \end{array} \right\}$$

Whence will arise these two Theorems for finding the Segments from the 3 Sides.

$$1. \frac{bb}{b} = a. \quad 2. \frac{cc}{b} = e.$$

$$\left. \begin{array}{l} 3. a:p :: b.c \\ c.p :: c.b \end{array} \right\} \text{Wherefore } \left\{ \begin{array}{l} ac = pb \\ cb = pc \end{array} \right\}$$

From whence these Theorems will arise for finding the Segments, the Sides, or the Perpendiculars.

$$1. \frac{pb}{c} = a. \quad 2. \frac{pc}{b} = e.$$

$$3. \frac{ac}{p} = b. \quad 4. \frac{eb}{p} = c.$$

$$5. \frac{ac}{b} = p. \quad 6. \frac{eb}{c} = p.$$

$$\left. \begin{array}{l} 4. b.p :: c.e \\ c.p :: b.a \end{array} \right\}$$

$$\text{Wherefore } \left\{ \begin{array}{l} be = pc \\ ca = pb \end{array} \right\}$$

I 3

And

And Consequently,

$$1. e = \frac{pc}{b}.$$

$$2. a = \frac{pb}{c}.$$

5. And since $e = \frac{pc}{b}$ and (by 3.) $\frac{cc}{b} = e$;

Therefore $\frac{pc}{b} = \frac{cc}{b}$. Wherefore $pcb = bcc$.

And dividing both by c , $pb = bc$.

That is, the Rectangle under the Legs, is equal to that of the Perpendicular into the Hypothenufe, &c. For, by proceeding after this Method, the Reader may easily discover many such Propositions as these: Which I leave to exercise his Skill and Diligence this way.

I. That the Rectangle under either Leg of a Right-angled Δ , and the opposite Segment of the Base, is equal to that under the Perpendicular into the other Leg.

II. The Square of the Hypothenufe is to that of either Leg :: As the Rectangle under the Hypothenufe, and the Segment of it, opposite to that Leg, is to the Square of the Perpendicular.

III. The Solid under the Perpendicular into the Rectangle of the Legs, is equal to that under the Hypothenufe into the Rectangle of its Segments.

IV. The Square of the Perpendicular is to the Square of any Leg, as the Segment opposite to the Leg, is to the whole Hypothenufe.

V. The

V. The Square of one Leg into the opposite Segment of the Hypothenuſe is equal to the Square of the other into its oppoſite Segment. Wherefore,

VI. The Squares of the Sides are as thoſe Segments.

56. If on the Hypothenuſe bc of a Rectangled Triangle, there be made a Semicircle bac , and on the other two Sides ab and ac , two more Semicircles bna and amc , that great Semicircle will be equal to the other two (by the laſt *Propoſition*.) And if from the greater Semicircle, and the two leſſer ones, you take away what is common to both; which are the two ſhaded Segments ab and ac ; what remains of each muſt be equal, *i. e.* the Triangle abc is equal to both the Lunes bna and amc .



And this is the Quadrature of the *Lunes of Hippocrates of Scio*.

57. When the Triangle bae is an *Iſoſceles*, then the Lunes will be equal, and then alſo the Triangle abo , being the half of abc , will be equal to each Lune. But if the Triangle be a *Scalene*, as in this Figure, the Lunes are unequal; and 'tis as difficult to divide the Triangle abc into two Parts by the Line ao , ſo as to be able to prove the Triangle abo to be equal to the Lune bna , and the Triangle oac to be equal to the other Lune amc ; this is, I ſay, as difficult as to find the Quadrature of the Circle.

N. B. Since this, ſeveral ways have been diſcover'd of ſquaring any aſſigned Portion of theſe Lunes (See the *Philofophical Tranſactions*, N^o. 259. pag. 4, 11.) Of which this is one.

Semi-right: Therefore the three Triangles EBG , EBF and EFG , must be each one the half a Square: And consequently, $GB:EB::\sqrt{2}:\sqrt{1}$; for the Square of GB is double the Square of EB ; and since similar Segments are as the Squares of their Chords, the Segments BG must be double of BE : Wherefore the half of one will be equal to all the other; that is, BDF equal to the Segment BE . And therefore the Rectilineal Triangle BEF , exceeding the Portion of the Lune by the half Segment BEF , and falling short of the Lune by the Segment BE , which is equal to that former half Segment BDF , the Triangle is exactly equal to the Portion of the Lune. Q. E. D.

And the Ground of all is this, that the Angle BCE being at the Center of one Circle, and at the Circumference of the other, must divide the Quadrantal Arch BGA , in the same Proportion as it doth the Semi-circular one BEA : On which depends the Equality of the Segments BE , and BDF .

And since the Triangle BCA is equal to the Lune L , (as is apparent by taking the common Segment $BGAB$, from the Semi-circle $BEAB$, and from the Quadrant $BGAC$) it will be easy to take from thence a Part, as the Triangle BOC , equal to the assigned Portion of the Lune. For having let fall a Perpendicular from E , to find the Point O , draw OC ; and then will the Triangle BOC be equal to the Triangle BEF , before proved equal to the Segment of the Lune. For the Triangles BCA and BEF are similar, as being each the half of a Square: And therefore the former to the latter will be as the Square of BA , to the Square of BE (6. 47.) their homologous Sides: That is, as BA is to BO (6. 25.) for BE is a mean Proportional between BA and BO . Farther, the Triangle BAC , having the same Height with BOC , will be to it as the Base AB to BO .

Where-

Wherefore the two Triangles BEF and BCO, being proved to have the same Ratio to one and the same thing, must be equal. Q. E. D.

And therefore to divide the Lune according to any given Ratio, you need only divide the Diameter AB, according to that Ratio in the Point O, and from thence erect a Perpendicular to find the Point E: Then draw EC, which shall cut off the assigned Portion of the Lune.

58. Two Chords cutting or crossing each other in a Circle, have the Segments *reciprocally Proportional*.



I say, that $ac : be :: de : ec$, and consequently the Rectangle aec is equal to the Rectangle deb .

For draw the prick'd Lines ab and dc , and the two Triangles abe and dce will be similar: Because, 1. The Vertical or Opposite Angles at e are equal (1. 23.) 2. The Angle b is equal to c , because standing both on the same Ark ad , and being in the same Segment (4. 12.) wherefore the two Triangles are similar, and consequently $ae : eb :: de : ec$. (6. 46.) Q. E. D.

59. If ac be the Diameter of a Circle, and bd a Perpendicular to it, de or be will be a mean Proportional between the



Segments of the Diameter ae and ec . Because de is equal to eb (by 4. 6.) and therefore since (by the last) the Rectangle bed (that is be Square) is equal to aec , as the Re-

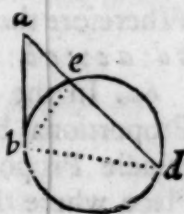
ctangles of the Parts of all crossing Chords are; the Line be or ed , must be a mean Proportional between ae and ec . Q. E. D.

60. Two

60. Two Lines ac and ad , drawn from a Point a , without a Circle, to the internal and opposite Part of its Circumference, are to each other *reciprocally* as their external Segments. I say, $ac, ad :: ae:ab$, and consequently the Rectangle cab is equal to dae . For supposing the Lines ce and bd to be drawn, the Triangles aec and adb will be similar, because the Angle a is common to both, and the Angle c is equal to d , because standing on the same Ark be (4. 12.) Wherefore $da:ab :: ca:ae$; and alternately, $da:ca :: ab:ae$; and by Inversion $ca:da :: ae:ab$ (6. 45.) And therefore the Rectangle cab is equal to dae . Q. E. D.

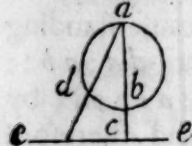
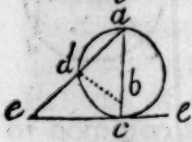


If one Line, as ab , touch a Circle, as in the Point b , and another Line ad , drawn from the same Point a , do cut it; then is ab (the Tangent) a mean Proportional between ad and ae (i. e. between the whole Secant, and the Part of it without the Circle.)



For drawing the Lines be and bd , the Triangles aeb and bad will be similar, because the Angle a is common to both, and the Angle abe (made by the Tangent, and Secant eb) is equal to d (an Angle in the opposite Segment) (4. 17.) therefore they are similar, and consequently ea (in the little Triangle) will be to $ab ::$ as that same ab is to ad , in the greater Triangle; i. e. $ea:ab :: ab:ad$. Q. E. D.

61. Let there be a Diameter ab cut in c by an infinite Perpendicular ee , whether within the Circle, as in *Fig. 1.* at the Circumference, as in *Fig. 2.* or without the Circle, as in *Fig. 3.* Let



there be drawn also from the Point a , any Right Line, as ae , cutting the Perpendicular in e , and the Circle in d . I say, it shall always be as $ad:ac::ab:ac$.

For drawing the Line bd , there will be made two Triangles that are similar, as $ea c$ and dab ; which will be so, because they have one Angle, as a , common to both, and the Angle d equal to c , because both are Right ones (for d is Right by 4. 14.) as being an Angle in a Semicircle, and c is Right by the Supposition. Wherefore the Triangles are similar, and consequently $ad:ac::ab:ae$. Q. E. D.

62. In the second Figure ad is always a mean Proportional between ae and ac ; in the first, the middle Proportional is aE , drawn from a , to the Place where the Line ec cuts the Circle.

63. If of a Triangle inscribed in a Circle, the Angle $b a c$ be bisected by the Line $a e d$,

I say, then $ba:ae::ad:ac$. For drawing the Line eb , there will be made two Triangles abd and aec , which are similar; because the Angle d is equal to c (4. 12.) as (being in the same Segment) or insisting on the same Ark, and bad is equal to $ea c$, by the Supposition. Wherefore the Triangles are



similar, and consequently $ba:ad::ae:ac$ (and therefore alternately $ba:ae::ad:ac$.) Q. E. D.

64. When

64. When the Angle at the Vertex is thus bisected, the Segments of the Base bc are also proportional to the Legs of the Triangle (*i. e.*) $be:ec::ba:ae$. For supposing ef drawn parallel to ba : Then will $ba:ac::ef:fc$ (6. 40.) But ef is equal to af , because the Angle aef is equal to eab , (*as being alternate Angles* 1. 31.) and consequently to caf (*by the Supposition*) wherefore the Triangle aef is an *Isoceles* (2. 15.) And therefore instead of putting of it as before, $ba:ac::ef:fc$, we may say $ba:ac::af:fc$. But as $af:fc::$ so is $be:ec$ (6. 42.) wherefore $ba:ac::be:ec$. Or, which is all one, $bc:ec::ba:ac$. Q. E. D.

N. B. *This Proposition is Universal; and if any Angle of a Triangle be bisected, the Legs about that Angle are proportional to the Segments of the opposite Side made by the Line bisecting the Angle.*

65. If two Circles touch one another (*in a Point within*) as a , and if to that Point you draw a Tangent and a Perpendicular acb (which will pass thro' both their Centers) (4. 5.) and if also you draw any Secant from the same Point, as aed ; I say, 'twill always be $ae:ad::ac:ab$. For having drawn the Lines ec and db , the Triangles aec and abd will be similar, as having the Angle at a common, and e and d both right ones; (by 4. 14.) and consequently $ae:ad::ac:ab$. Q. E. D.

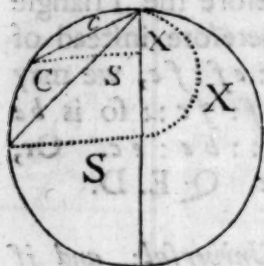


66. The Ark ec is to the Ark db , as the whole Circle aec , to the whole Circle adb . (6. 49. and 4. 11.)

P R O P.

P R O P.

67. If two (or more) Chords, as c , C , issue from the same End of any Diameter of a Circle; Their Squares shall be directly, as the Versed Signs x X .



And shall also be equal to the Rectangles under the Diameter and such Versed Sines.

Let fall the right Sines s and S . Then will $cc = ss + xx$, and $CC = SS + XX$; and if the Diameter be called D , its Parts will be x and $D - x$, X and $D - X$:

But $ss = x$, $D - x$, and $SS = DX - XX$ (by 66. of this Book) wherefore substitute those two last Quantities instead of the Equals ss and SS ; and you will have $cc = Dx - xx + xx$ (that is) Dx and $CC = DX - XX + XX$ (that is) DX ; which proves the latter Part of the Proposition, that the Square of the Chord is always equal to the Rectangle under the corresponding versed Sine, and the whole Diameter.

And 'tis plain, that

$$Dx.DX :: x.X. \text{ Q. E. D.}$$

P R O P.

68. A Circle, whose Area is equal to the Convex Surface of a given Cone, will have its Radius a mean Proportional between the Side of the Cone and Radius of its Base.

Let

Let the Side of the Cone be $= a$, the Radius of the Base $= r$; then the Diameter will be $2r$, and the Periphery $= 2re = c$. But half the Periphery into the Side of the Cone is $=$ to the Convex Surface of the Cone (by....) that is, are expresses the Area of the Cone. Now since $\sqrt{ar}$ is a mean Proportional between a and r (for $a : \sqrt{ar} :: \sqrt{ar} : r$) I imagine $\sqrt{ar}$ to be the Radius of the Circle whose Area $=$ Area of the Cone. Then will its Diameters be $2\sqrt{ar}$, and its Periphery $2\sqrt{are}$; and by Multiplication of $2\sqrt{are}$, the Periphery into $\frac{1}{2}\sqrt{ra}$ the half Radius; or $\sqrt{are}$ into $\sqrt{ra}$, the Radius of the Circle will be $are = b$, the Surface of the Cone. Q. E. D.

69. The Convex Surface of a right Cone is to the Area of its Base $::$ as the Side of the Cone is to the Radius of the Base.

For since the Convex Surface of the Cone (by what is said after 14. Book 4.) is equal to a Triangle whose Base is equal to the Periphery of the Circular Base of the Cone, and its Height the Side of that Cone, call the Periphery c , and the Side of the Cone a , then will $\frac{ac}{2}$ express the Area of the Convex Surface, and the Area of the Base will be $\frac{rc}{2}$. (by Art. 26. Book 4.) But there is no Doubt

but $\frac{ac}{2} : \frac{rc}{2} :: ar$. Wherefore, &c.

70. A Circle whose Radius is equal to the Diameter of the Sphere, will have its Area equal to the Sphere's Surface.

Let

Let the Radius of such a Circle be $2r$, then its Diameter is $4r$, and its Periphery will be $4\pi r$; and by multiplying that by $r = \text{half of Radius}$, the Area is $4\pi r^2$. Let then the Radius of the Sphere be r , then will its Diameter be $2r$, and the Periphery of a great Circle $2\pi r$, which being multiplied by the Radius r , makes $2\pi r^2$; the half of which is πr^2 , the Area of a great Circle; but the Area of 4 such Circles is equal to the Sphere's Surface (by *Cor. V. p. 76.*) that is, $4\pi r^2 = \text{to the Sphere's Surface}$; which was above proved equal to the Area of the Circle, whose Radius was equal to the Sphere's Diameter. Wherefore, &c.





ELEMENTS

OF

GEOMETRY.

BOOK VII.

Of Incommensurables.



Lesser Quantity is said to *measure* a greater, when being taken a certain number of Times, it is exactly equal to the greater. *V. gr.* Suppose a Fathom to contain six Feet; then may one Foot be said to *measure* that Fathom, because being taken or repeated six times, it will be exactly equal to the Fathom.

2. The Quantity which is thus a *Measure* to a greater Quantity, is called a *Part* of that greater; and the greater Quantity is call'd the *Multiple* of the lesser. So a Foot is the *Part* of a Fathom, and a Fathom is the *Multiple* of a Foot.

3. If you take the Quantity (*of a common French Pace*) which is two Foot and half, and try with that

to measure a Fathom, you cannot do it: Because if you add that Pace only twice, it will make but five Foot, which are less than the Fathom; and if you take it three times, it makes seven Foot and half, which are more than the Fathom; so that this Quantity of two Foot and half cannot measure the Fathom, and therefore properly speaking is not a *Part* of it. But nevertheless they may be said to be *Parts* of the Fathom, because this Quantity contains five half Feet; for an half Foot is a *Part* of a Fathom, because being taken 12 times, it will just measure it; so therefore this Pace contains *Parts* of the Fathom, because it contains five half Feet, which are $\frac{5}{12}$, that is five twelfths of a Fathom.

4. When two Quantities are such, that a third can be found which shall be an (*Aliquot or Even*) Part of both, that is, which shall *measure* them both exactly: Then those Quantities are said to be commensurable: As for Instance, a Pace and a Fathom are two commensurable Quantities, because we can find a third Quantity, *viz.* half a Foot, which will measure them both; for if the half Foot be taken five times, it makes the Pace, and taken 12 times, it makes the Fathom.

5. But when it is not possible to find any third Quantity which can measure two others, then those two Quantities are called Incommensurables.

6. Commensurable Quantities are as *Number to Number*, that is, those Quantities can be expressed by Numbers, so that as one Quantity is to the other, so shall one Number be to the other. Thus a Line of six Foot or a Fathom, and a Line of two Foot and a half, as a Pace, are to one another as Number to Number. For half a Foot measuring them both, the latter by being taken five times, and the former by being taken 12 times, it's plain that one Line contains 5 half Feet, and the other 12, and therefore they are as 5 to 12, or as Number to Number. 7. If

7. If two Quantities are not as Number to Number, that is, if it be impossible to express their Magnitudes by two Numbers, they are Incommensurable: As is plain from the last.

8. We ought then to see whether there are in Reality any such Quantities whose Magnitude cannot be express'd by Numbers; and if there be any such, we must say that there are *Incommensurable Quantities*.

9. A *plane Number* is that which may be produced by the Multiplication of two Numbers (*one into another*) *v. g.* 6 is a plane Number, because it may be produced by the Multiplication of 3 by 2: For twice 3 makes 6. So also 15 is a plane Number, arising from 5 being multiplied by 3; and 9 is a plane Number, produced by the Multiplication of 3 by 3.

10. Those Numbers which, being multiplied one by another, do produce a plane Number, are called the *Sides* of that Plane, as 2, and 3 are the Sides of the Plane 6; and 3 and 5 are the Sides of 15.

11. If we imagine Units to be little Squares, those Squares may be formed into a Rectangle, if their Number be a Plane. *V. gr.* 12 Squares may be placed in the form of a Rectangle, one of whose Sides may be 6 and the other 2, and 48 will make a Rectangle whose two Sides may be 12 and 4. See the following Figures B and C.

12. A *square Number* is a Plane, whose Sides are equal; as 4 arising from the Multiplication of 2 by 2; as 9, the Product of 3 by 3: And 16 made by 4 multiplied by 4, &c.

13. A square Number may be ranged into the form of a Square, and that Number which can be ranged into the Form of a Square, is a square Number, and that which cannot be ranged into the form of a Square, is not a square Number.

14. *Similar plane Numbers* are those which may be ranged into the Form of



be ranged into the Form of similar Rectangles; that is, into Rectangles, whose Sides are proportional; such are 12 and 48; for the Sides of 12 are 6 and 2 (*See Fig. B*) and the Sides of 48 are 12

and 4 (*See Fig. C*) But $6:2::12:4$, and therefore those Numbers are similar.

15. All square Numbers are similar Planes (6. 32.)



16. Every Number may be placed in the Form of a Right Line, and in that Disposition may be taken for a Plane. Thus 3

(in *Fig. A*) may be conceived as a Plane similar to 12 or B; For the Sides of the Plane 3, are 1 and 3, (because once 3 is 3) and the Sides of 12 are 2 and 6. But as $1:3::2:6$.

17. There are Numbers which are not *similar Planes*: As if you examine from 1 to 10, you will find indeed that 1, 4, 9, being Squares are similar, and so are 2 and 8, which have one Side double to the other. But the rest, as 3, 5, 6, 7, are by no means similar Planes.

18. If one square Number be multiplied by another, the Product will be a third square Number.



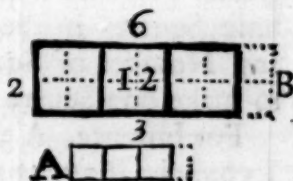
Thus A 4, and B 9, being both Squares, do, when multiplied into one another, produce the Number 36 or C: And I say that third Number is a Square. For the Meaning of multiplying B by A, is take B as often as there are Units in A.

But

But I may consider the whole Number B 9, as one only Square, and I can take that as often as there are Units or little Squares in A. And as the Units in A are ranged into a Square, so I can range the Square B as often into a square Form, just as if it were an Unit. So that there will be four such Squares of B, which, being placed as you see in the Figure, will make the Square C or 36.

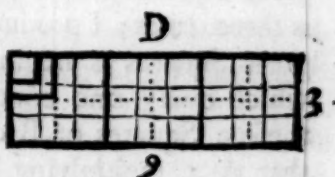
19. If two Numbers are similar Planes, the greater may be divided into as many Squares as there are Units in the lesser. A, 3. and B,

12. are similar Planes; so that the Side 3. is to 6 :: as the Side 1. is to 2. Wherefore I can divide the Plane B, 12 into 3 Squares placed just in such a manner as those 3 little Squares in the Plane A. And every one



of the great Squares of B shall answer to 4 of those in A. So also if the Planes

had been 8 and 72; I can divide 72 into eight Squares, of which every one shall contain 9 of those in the lesser Plane 8. The same would come to pass also, if



either one, or both the Numbers had been Fractions. As if A contain 3 and $\frac{1}{2}$, and B 14. I can divide 14 into three Squares and half, disposed just like those in A; as may be seen by the Partitions in the Figure, and by the half Square added in prick'd Lines.

In like Manner if the Planes were B 12, and D 27, I can divide 27, not only into three Squares, disposed after the same Manner as those in A: But also into 12 Squares, so ranged as those in B, as the prick'd Lines in the Figure D do shew. The Way to do which is to divide the Sides of the greater Plane into as many Parts as the homologous Sides of the lesser Plane

are divided into; the Figure shews the thing, and makes it easy.

20. Those plane Numbers which can be so divided, as that there are as many Squares in the greater Plane, as there are Units in the lesser, are similar; this is the Converse of the former.

21. Two similar plane Numbers, multiplied one into another, do produce a Square. For having divided the greater Plane into as many Squares as there are Units in the lesser (7. 19.) one Plane will be multiplied by the other, if the greater Squares of the greater Plane be taken as often as there are Units or little Squares in the lesser Plane: But to multiply any Number of Squares by the same Number, is to make one Square out of all those Squares.

For Instance, A 3. and B 27. being similar Planes, I consider B 27. as a Plane compos'd of three great Squares, as A 3. is a Plane compos'd of three Units, or three little Squares. So that if I take all these three great Squares as often as there are Units in A, that is three times; I produce then three times three such great Squares as are in B, that is 9 such squares; of which every one contains 9 of those in A, and all these 9 Squares of B contain 81 of those of A; so that A 3. multiplying B 27. produces 81. which is a



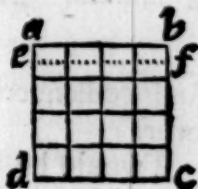
Number of the lesser Squares rang'd into a Square Figure; and by consequence a square Number (7. 13.) In like Manner if the Planes were B 12. and D 27. I divide 27 into 12 Squares, which I multiply by 12. and there are produced 144 greater Squares ranged in the Form of a Square, which do contain in all 324

of those of the lesser Plane. (N. B. To divide 27 into 12 Squares, each Square must be 2, 25. (or two and a Quarter) as you may see it is in the Figure D. N^o 19.)

22. If two Plane Numbers are similar, after what Form soever you range one, the other also may be so disposed. Let 3 and 12 be similar Planes. If 12 be so rang'd in a Right Line that will make a Rectangle, one of whose Sides shall be 12, and the other 1: I say that 3 may be so disposed as to make a similar Rectangle, one of whose Sides will be 6, and the other the half of one, &c.

23. If one Number divide another that is a square one, a third shall be produced which will be a Plane similar to the Divisor.

Let there be a Square ac 16, and let it be divided by any Number, as suppose by 8, which is done if you take the eighth Part of the Side ad , viz. ae , and thro' e draw the Parallel ef : For by that means you will have the Plane af , which will be the eighth Part of the Square ac . But to divide a Number or a Plane by 8, is to take the eighth Part of that Number or Plane.



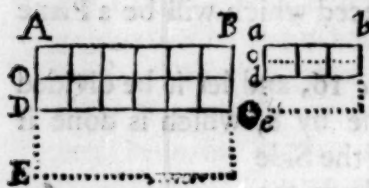
I say the Plane af is similar to 8; for 8 being ranged into a Right Line, so as to make a Rectangle, one of whose Sides shall be 8, and the other 1, shall be similar to it, because ae was taken the eighth Part of ad or ab : Wherefore as $8 : 1 ::$ (which are the Sides of the Plane 8 the Divisor) so shall $ab : ae$ (which are the Sides of the Plane of the Quotient arising when the Square ac was divided by 8. Therefore if one Number divide another that is a Square, &c. Q. E. D.

24. If two Planes multiplying one another do produce a Square, those Planes are similar.

25. Two Plane Numbers which are not similar, if they are multiplied into one another, cannot produce a Square. These two Propositions are Confectaries from the foregoing ones.

26. If two Numbers are similar Planes, their *Equi-multiples* and any of their (*respectively*) equal Parts, are also similar Planes. Let the Planes be $abcd$. 3. and $ABCD$. 12. so that $ab:AB::bc:BC$. I say, if you take the double of the one, and the double of the other (or any other *Equi-multiple*, be it what you please) those doubles shall be similar.

For having taken ae double to ad , and AE dou-



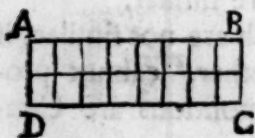
ble to AD : in order to make the Plane be double to bd , and BE double to BD : 'Tis clear that $ad:AD::ae:AE$. But $ad:AD::ab:AB$. Wherefore also $ae:AE::ab:AB$.

And consequently the Planes be and BE are similar.

'Twould be the same thing had you taken their Halves bo and BO , or any other equal Parts of each.

27. If two Numbers are not similar Planes, their *Equi-multiples*, and all their (*respectively*) equal Parts will also be not similar, which follows from the last.

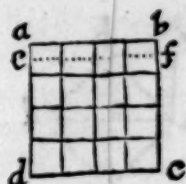
28. Between any two similar plane Numbers whatsoever, there is to be found a mean Proportional. Let the two Numbers be 2 and 8, I say it is possible to find a Number which shall be a mean Proportional between them. For if we imagine the Plane 8 to be ranged in a Right Line AB , and the Plane 2 also be ranged in another Right Line, as AD , and that out of those two Right Lines there be



formed the Plane AC , 16. That Plane AC , 16. will be produced by the Multiplication of the two Numbers 2 and 8 (6. 17, and the following Pro-

positions) and consequently the Number of the lit-
tle

the Squares of the whole Plane AC, 16. shall be a square Number (7. 21.) and they may be ranged into the Form of a Square (7. 13.) Let them then be disposed into the Square ac . So shall the Square ac be equal to the Plane AC; for 'tis only the same Number dispos'd or rang'd after another Manner. Wherefore (6. 59.) the Side ab 4 shall be a mean Proportional between AD 2, and AB 8.



29. Between two Numbers non-similar a mean Proportional can't be found. Let the Numbers be 4, and 6. Range each of them into a Right Line, and multiply them, they will produce the Plane 24. But this Plane 24 is not a square Number (7. 25.) and consequently cannot be ranged into a square Form. Wherefore 'tis impossible to have any *Mean* between 4, and 6. For such a pretended mean Proportional must, multiplied by itself, produce a Square, which (as hath been prov'd elsewhere) will be equal to the Plane made between 4 and 6. (6. 59.) which is impossible, because this Plane 24, made out of 4 and 6, is not a square Number.

30. Let there be two Lines ae and ec , so to one another, as one Number to another non-similar. *V. gr.* as 1. to 2. Let also eb be a mean Proportional, so that $ae:eb::eb:ec$. I say, that eb is *incommensurable* with the two Extremes ae and ec . For ae and ec , being as 1 to 2, (*i. e.*) as Numbers non-similar (by the Supposition) as also are all their Equimultiples (7. 27.) 'tis impossible to find a mean Proportional between ae and ec (by the Precedent) and consequently eb cannot be to ae , or to ec , as Number to Number. Wherefore it is *incommensurable* with them.



31. The Diameter of a Square ab is Incommensurable to the Side ac . For taking ad double to ac , and making the Triangle abd , it shall be similar to the Triangle abc ; because cd being equal to cb , the Angle d is equal to cbd (2. 15.) and the Angle d must be a Semi-right one as well as cab ; wherefore abd is a right Angle; and consequently $ac : ab :: ab : ad$. That is, ab is a mean Proportional between ac 1. and ad 2. and therefore Incommensurable (*by the Precedent.*)



COROLLARY.

Hence 'tis impossible to express one Square that shall be Double of another in rational Numbers.

32. The Power of a Line is the Square which is made upon it. Thus the Power of the Line ac (*Fig. preced.*) is the Square $acbc$; and the Power of the Line ab is the Square $abdf$. And we say that Line ab is double in Power (in Latin *bis potest*) to the Line ac , which is a manner of speaking borrowed from the *Greeks*, and generally receiv'd amongst *Geometers*.

33. The Diameter ab is *Commensurable in Power* to the Side ac : That is, its Square $abdf$ is Commensurable to the Square $acbc$, for 'tis indeed double to it.

34. But if you take ao , a mean Proportional between ab and ac , that mean ao shall be Incommensurable to them even in Power; *i. e.* the Square of ao is Incommensurable to the Square of ab , or to the Square of ac , for the Square of ac $a — c$ to the Square of ab , is in a Duplicate Ratio of ac to ao (6. 22); that is, as ac to ab 6. 30. But ac is Incommensurable to ab (7. 31.) wherefore the Square of ao is Incommensurable to the Square of ac .

35. There is a *Second Power* of a Line which is called the Cube, which is made by multiplying the Square by that first Line, or Root.

36. If two mean Proportionals, an and am , be taken between ac and ab ; so that ac : $an :: am : ab$; the Line an will be Incommensurable in this second Power to ac , (*i. e.*) The Cube of ac will be Incommensurable to the Cube of an , because the Cube of ac to the Cube of an is in a *Triplicate Ratio* of the Side ac , to the Side an ; *i. e.* as ac to ab . But ac and ab are Incommensurable; wherefore, &c. However, ac and ab are Commensurable in the second Power; for the Cube of ab is double to the Cube of ac .

37. 'Tis easy to apply to *Solid Numbers* what hath here been said of Plane ones: And those are called *Solid Numbers*, which arise from the Multiplication of a Plane Number by any other whatsoever. *V. gr.* 18 is a solid Number made of 6 (which is a Plane) multiplied by 3; or of 9 multiplied by 2.

38. *Similar Solid Numbers* are those, whose little Cubes may be so ranged, as to make similar and rectangular Parallelopipeds.

39. *Cubick Numbers* are such as can be ranged into the Form of Cubes, as 8. or 27, whose Sides are 2 and 3, and their Bases 4 and 9.

40. Every

40. Every cubick Number, multiplying another cubick Number, produces a third cubick Number.

41. Between two similar solid Numbers there may be found two mean Proportionals.

That which hath been demonstrated, in respect to Plane Numbers, may be applied to Solids.

42. These Demonstrations, by which 'tis proved that there are incommensurable Lines and Magnitudes, shew also that a *Continuum* is not compos'd of finite Points: For if the Diameter as well as the Side of a Square were compos'd of finite Points, a Point would measure both the Side and the Diameter, for that Point would be found a certain Number of Times in the Side, and another determinate Number of Times in the Diameter, which the preceding Propositions prove impossible.

43. Because in a Rectangle Triangle the Square of the Hypotenuse is equal to the Sum of the Squares of the Legs; (6. 61.) we have always used this Triangle for the Discovery of Incommensurables. For if all the three Sides are commensurable, they may all three be express'd by three Numbers, and then the Square of the greatest Number will be equal to the Sum of the Squares of the other two. As if the greatest Side or Hypotenuse be 5 Feet, the least Side 3, and the middle one 4: The Square of 5 will be 25, the Square of 3, 9, and the Square of 4 will be 16: And 9 and 16 added together do make the great Square 25. But if the least Side of such a Triangle be 2, and the middle one 3, then the greatest Side cannot be express'd in Numbers, because the Square of the least Side 4, added to the Square of the middle Side 9, makes 13, which express the Square of the greatest Side. But as that Number 13 is not a square Number, so its Side or Root cannot be express'd by any Number.

44. At all times Men have been solicitous to find out some Method of discovering proper Numbers to express

express the three Sides of a Rectangle Triangle, so as to be assured that all the three Sides are Commensurable. Therefore I here shew you such a Method, by which you may find out all the possible Numbers that are proper for this Purpose.

45. If you take any two Numbers (even Unity itself) differing but by an Unit, and add the Squares of them together, the Sum will be a Number which shall be the Root of a Square equal to two Squares; and that Number will express the greatest Side of a Rectangle Triangle, whose middle Side shall be that Number lessen'd by Unity, and the least Side shall be the Sum of the two first Numbers. *V. gr.* Having taken 1 and 2, and squared each of them, you have 1 and 4; add those two Squares together, and the Sum is 5. I say 5 will express the greatest Side, and then 4 will be the middle one, and 3 the least; and 25, the Square of the Hypothenuse, will be equal to the Sum of the other two Squares. In like manner if you take 2 and 3, and add the Squares 4 and 9 together, the Sum is 13. Then I say, will 13, 12 and 5 be three Sides of a Rectangle Triangle; so that 169, the Square of 13, shall be equal to 144, and 25, the Squares of 12 and 5. Moreover if you take 3 and 4; the Sum of their Squares 9 and 16, makes 25; wherefore I say 25 may be the greatest Side of a Rectangle Triangle, whereof 24 will be the middle Side, and 7 the least Side.

It must be observ'd also, that the Equimultiples of any 3 Numbers thus found, will do the same thing: Thus, having found 5, 4 and 3, their doubles 10, 8 and 6, will represent the three Sides of a Rectangle Triangle, so that 100, the Square of 10, shall be equal to the Sum of 64, and 36 the two Squares of 8 and 6. And their Triples also 15, 12 and 9, will do the same thing: For any one may see that all these Numbers, still having the same Proportion, do as it were

were constitute but one only Triangle, viz. that which is express'd by 5, 4, and 3. And therefore all those Numbers may be taken for the same.

N. B. *The three Sides of a Rectangled Triangle will then only be commensurable, when they are in this Proportion, viz. as $aa+ee$, $aa-ee$, and $2ae$. That is, the Sum of two Square Numbers, the Difference of their Squares, and the double Rectangle of their Roots.*





ELEMENTS OF GEOMETRY.



BOOK VIII.

Of Progressions and Logarithms.

1.  *Progression* is a *Series* or Rank of Quantities which keep between one another any kind of similar Relation or Proportion; and every one of the Quantities is called a *Term*.

2. When the Terms which follow one another do equally increase or decrease, the Progression is called *Arithmetical*; as are all Numbers proceeding according to the natural Order of the Figures, as 1, 2, 3, 4, 5, 6, &c. As also all odd Numbers, as 1, 3, 5, 7, 9, 11, &c. or as 4, 8, 12, 16. or as 20, 15, 10, and the like.

3. Arithmetical Progression may be increased infinitely, but not diminished.

4. If

4. If in an Arithmetical Progression there be four Terms, the Difference between the two first of which is equal to the Difference between the other two, those four Terms are said to be *Arithmetically Proportional*: As in the Progression of the natural Numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, &c. If you take four as 2, 3 :: 9, 10, (*This Mark :: I shall for the future use to signify Arithmetical Proportion*) there will be the same Arithmetical Proportion between 2 and 3, as there is between 9 and 10; that is, 10 exceeds 9, as much as 3 doth 2: So also 3 : 5 :: 8 : 10, are in Arithmetical Proportion; and so are 1 : 5 :: 5 : 9, where 5 being taken twice, is an Arithmetical mean Proportional between 1 and 9.

5. In Arithmetical Proportion the Aggregate or Sum of two Extrems is equal to the Aggregate of the two Means, as in 2 : 3 :: 9 : 10. the Sum of 2 and 10 is equal to the Sum of 3 and 9, that is 12; so also in 3 : 5 :: 8 : 10. the Aggregate of 3 and 10 is 13, which is equal to the Aggregate of 5 and 8. And the Reason of this is self-evident: For tho' 10 exceeds 8, yet that which is added to 8, (*viz.* 5.) doth just as much exceed 3, which is added to 10, and so there necessarily arises an Equality between them.

6. The Sum of the first and last Terms in any Arithmetical Proportion, is equal to the Sum of the second and the last save one; or to the Sum of the *third from the first Term, added to the third, accounted backward from the last*, &c. as in the first Example, 1 and 9 make 10, and so do 2 and 8, 3 and 7, or 6 and 4 always make 10. And in the middle remains 5, which being taken twice (as if it were equivalent to the Terms, because 'tis equally distant from the first and last Term) makes also 10.

7. If you add the first Term to the last, and multiply that Sum by half the Number of the Terms, the Product shall be equal to the Aggregate or Sum
of

of all the Terms. As in the former Example, 1 added to 9, makes 10, and 10 multiplied by $4\frac{1}{2}$ (or 4, 5.) for there are 9 Terms, produces 45, which is the Sum of all the Terms from 1 to 9. As is manifest from the Precedent.

8. When the Terms of the Progression are continual Proportionals; that is, when the first is to the second, as that is to the third Term, as the third is to the fourth, and as the fourth is to the fifth, &c. then the Progression is call'd *Geometrical*, as 1, 2, 4, 8, 16, 32 :: Or as 1, 3, 9, 27, 81 :: Or again, as 3, 12, 48, 192, 768; Or *descending*, as 8, 4, 2, 1 :: Or lastly, as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, &c.

9. Geometrical Progression may be encreas'd and diminish'd infinitely.

10. When the Progression begins with 1, the second Term is called the *Root*, *Side*, or first *Power*; the *third* is called the *Square* or second *Power*; the *fourth*, the *Cube* or third *Power*; the *fifth*, the *Biquadrate* or fourth *Power*; the *sixth*, the *Sur-solid* or fifth *Power*; the *seventh*, the *Quadrato-Cube* or sixth *Power*, &c.

11. If (in such a Progression) you take four Terms, the former two of which are as much distant from each other, as the two latter are: Those are simply Proportional, and the Rectangle of their Extreame is equal to that of their two middle Terms.

12. Let the Quantity AB be so divided in C, D, E and F, &c. that $AB:AC::AC:AD::AD:AE$, &c. Then I say, $BC:CD::DE:EF$, &c. are in continual Geometrical Proportion; and also that $AB:AC::BC:CD::CD:DE$, &c. for because $AB:AC::AC:AD$, it will follow, by Division of Proportion, that AB less AC : (that is CB): AC :: as AC less AD : (that is DC) AD , and consequently alternately $CB:CD::AC:AD$, or as $AB:AC$, and so of all others it may be proved :: $DC:ED::EF:GF$, &c.

L

13. Let

13. Let there be a Progression of Quantities in a Right-line BC, CD, DE, EF, &c. let Cd be equal to the *second* Term CD, that so we may have Bd the Difference between the *first* and *second* Terms: And let it be made as Bd : BC::BC to a fourth Line, viz. BA. I say, that if the Number of the Terms BC:CD:DE, &c. be finite, though ever so great, all those Terms taken together, although there be an hundred thousand Millions of them, shall be less than BA. But if we suppose the Progression infinite, or that the Terms are infinitely many, then shall all of them taken together be exactly equal to BA. For since by the Supposition Bd , (that is BC less Cd or CD) is to BC::BC, (that is AB less AC) : AB, it may easily be found that as BC:CD::AB:AC::AC:AD, &c. and consequently all the Terms CD, DE, EF, &c. will always be found within, or be hither the Point A. To which it approaches the nearer, the more the Number of the Terms is increas'd. So that we see plainly, that all these Terms (which in Books are usually call'd *Parts Proportional*) tho' they be actually infinite, cannot make an infinite Length, because they will be all included within the Line BA.

14. This Demonstration will appear much more easy and sensible by the Example of a particular Progression, where the Terms are in a double *Ratio*; v. gr. Let CB be double to DC, and DC double to DE, &c. For if the Number of the Terms be here finite, tho' it be an hundred thousand Millions, and you take the last and least Term, for Example FE, and add to it another Quantity, as suppose AF, equal to it: It is then plain, that EA must be equal to the Term ED, which is the last save one: For ED is double EF by the Supposition (the *Ratio* being

being every where double) and EA is also double to EF by the Construction, it having been made so, by taking FA equal EF . In like manner AE with DE , that is AD , shall be equal to the following Term CD , and at last AC will be equal to BC . So that from hence it appears, that the first or greatest Term is always equal to all the others taken together, provided there be added to them but a Quantity equal to the last and least Term; but if nothing be added, the first Term is always greater than the Sum of them all.

If these Terms are suppos'd to be actually infinite, then the greatest BC will be exactly equal to all those infinite others taken together CD , DE , EF , &c. For any one may easily discern, that the more there are of such Terms, the more you approach towards A , by cutting off still the half of the Remainder: But when any Quantity is thus lessen'd by half, and the Remainder again by half, and then the half of that third Remainder taken, and so on: 'Tis plain, that by supposing the Diminution to be made an infinite number of Times, nothing at last will remain.

This also might be demonstrated by a *Reduction ad Impossibile*, by shewing that all those infinite Terms, taken together, are neither greater nor less than AB .

15. Hence may the Difficulties rais'd by the Schoolmen against the (*Infinite*) Divisibility of a *Continuum* be solved, tho' to Persons ignorant of *Geometry*, they appear unsolvable: But indeed at the Bottom they are nothing but meer Paralogisms.

16. If two Progressions are supposed, one Geometrical, beginning with 1, and the other Arithmetical, beginning with 0, so that the Terms in one shall be placed over, and answer respectively to those in the

other; the Arithmetical ones are called *Logarithms*, *Exponents*, (or *Indexes*) as in the following Ranks.

0.	1.	2.	3.	4.	5.	6.	7.	8.
1.	2.	4.	8.	16.	32.	64.	128.	256.

17. That which is produced in a Geometrical Progression by Multiplication and Division, is effected in the *Logarithms* by Addition and Subtraction: As, if having three Numbers given; $2:8::64$, you would find a fourth Proportional to them in Geometrical Progression; you must multiply 64 by 8, (which are the two middle Terms). For the Product 512 shall be equal to the Product made by 2, and the fourth Number sought, they being the two Extreams of four Proportionals. And to find this fourth Number, you need only divide 512 by 2, and the Quotient will be 256. So that $2:8::64:256$, and 64 and 256 will be just as far distant from one another in the Order of the Progression, as 2 and 8 are.

But if instead of the Geometrick Numbers $2:8::64$, you had used their Logarithms $1:3::6$, which answer to them in the Progression, and were minded to find a fourth Logarithm, then you must have added 3 and 6, which makes 9, and from thence having subtracted 1, there would remain 8. The Logarithm answering to the Geometrick Number 256.

18. So also, if there be two Geometrick Numbers 4 and 8, to which the Logarithms 2 and 3 do answer; by multiplying 4 by 8, you produce 32; the Number under the Logarithm 5, which is the Sum of the Logarithms of 2 and 3.

19. In like manner by multiplying 16 by itself, there will be produced 256, which stands under the Logarithm of 8, the Sum of 4 added to itself.

20. So if the Geometrical Number were required that shall answer to, or stand under, the Logarithm 16, you must take 256, which stands under 8, and multiply it by itself, and it will produce 65536, the Number required.

21. If moreover the Geometrical Number answering to the Logarithm 23 were required, you may take any two Logarithms, whose Sum is 23, as suppose 7 and 16, and multiplying the Geometrical Numbers under them, *viz.* 128 and 65536 one by another, the Product will be 8388608. The Number which ought to stand under the Logarithm 23, or in the 23d Place of a Series of Geometrical Proportionals, beginning from 1.

22. From hence appears the Way of answering that ordinary Question, How much a Horse would cost, if bought on this Condition: That for the first Nail in his Shoe a Farthing were to be paid, for the second Nail two Farthings, for the third Nail four Farthings, for the fourth Nail eight Farthings, and so on, still doubling for 24 times: For the 23d Place in such a Progression would be the last Number 8388608 Farthings, which, being reduced, is 8738*l.* 2*s.* 8*d.* and being doubled according to (8. 14.) gives the whole Price of the Horse 17476*l.* 5*s.* 4*d.*

23. Where two compleat Progressions are fitted so as to answer one to another, the Geometrical to the Arithmetical; as suppose in Tables for that purpose calculated in Books, there abundance of Pains and Labour is spared, in finding the Geometrical Numbers: For Instance, let those three Numbers 32, 64, 128 be given, and that a fourth Proportional were required: Instead of multiplying 64 by 128, and dividing the Product by 32, (which Way is very tedious in great Numbers) you need only take the Logarithms of the three given Numbers, *viz.* 32, 64,

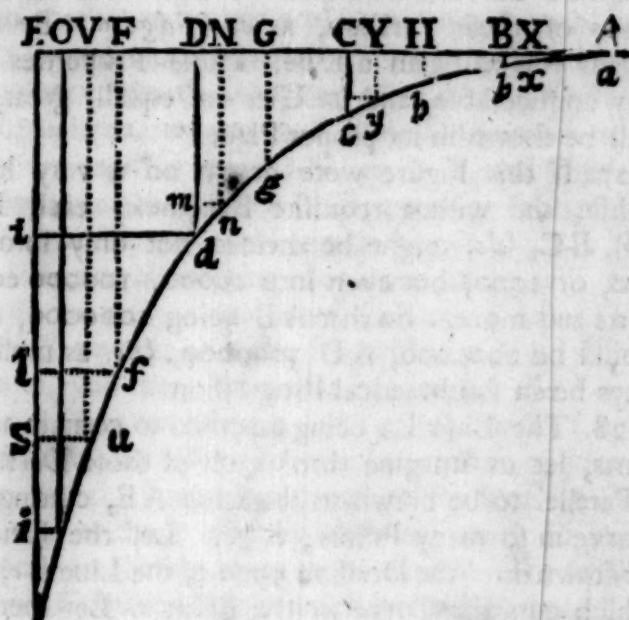
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128;

128; and adding the 2d and 3d together, from their Sum subtract the first, the Difference will be the Logarithm of the corresponding Geometrick Number 256.

24. But because in such a Geometrical Progression all Numbers will not be found, this Medium hath been discovered; they have calculated two Progressions, one of which contains all Numbers 1, 2, 3, 4, 5, 6, 7, 8, &c. which seems to be an Arithmetical Progression, but yet hath in reality the Properties of a Geometrical one. And the other, which contains Numbers in Appearance the most irregular, is nevertheless a true Arithmetical Progression. See here a Line, which will discover perfectly all these Mysteries.

25. Let the Right-line AE be divided into the equal Parts AB, BC, CD, DE, &c. from the Points A, B, C, D, E, &c. let the Lines Aa, Bb, Cc, Dd, and Ee, be drawn all (*perpendicular to AE, and consequently*) parallel to one another: And let them be all in a Geometrical Progression: As let Aa be 1, Bb 10, Cc 100, Dd 1000,



Ee 10000, &c. Then shall we have two Progressions of Lines, the one Arithmetical, and the other Geometrical: For the Lines AB, AC, AD, AE, are in Arithmetical Progression, or as 1, 2, 3, 4, 5, &c. and so do represent the Logarithms; to which the Geometrical Lines Aa, Bb, Cc, &c. do correspond.

26. Let each of the equal Parts ED, DC, CB, &c. be divided equally again in F, G, H, and let the Parallels Ff, Gg, &c. be drawn, and be mean Proportionals between the Collateral ones; that is, $Ee : Ff :: Ff : Dd :: Dd : Gg$. Let there also be more mean Proportionals drawn from the middle of each Sub-division EF, FD, DG, and so on, till these Parallel Lines growing very numerous, have at last but a very small Distance from each other; then imagine a Curve Line drawn through all the Extremities of these Parallels, as *confdgba*: By this Means you will gain a Line, whose Properties are very considerable, and its Uses are equally great, as shall be shewn in its proper Place.

27. If this Figure were drawn on a very large Table, and with a requisite Exactness, each Part AB, BC, &c. might be divided not only into an 100, or 1000, but even into 10000, 100000 equal Parts and more. So that AB being 1000000, AC would be 2000000, AD 3000000, &c. as must always be an Arithmerical Progression.

28. The Line Ee being supposed to contain 1000 Parts, let us imagine thro' each of those Divisions a Parallel to be drawn to the Line AE, cutting the Curve in so many Points; *v. gr.* Let the Line io be drawn thro' the Division 9900 of the Line Ee, and which cuts the Curve in the Point o. Let there be also supposed the Parallel (to Ee) Oo, cutting the Line AE in the Division 399563. Then any one may know that 399563 is the Logarithm of the Number 90000. In like manner if Su passed thro' the Division 9000 of the Line Ee, and the Line uv were drawn cutting AE in 395424, then would that Line uv be the Logarithm of 9000, &c.

29. So that by this means a Table of Logarithms from 1 to 10,000 may easily be made, and farther, by producing the Line AE.

30. Note,

30. Note, to obtain all the Logarithms from 1 to 10000; 'twill be enough to seek the Logarithms from 1000 to 10000: That is (having drawn the Parallel $d t$) to take the Logarithms of all the Divisions from t to e , which Logarithms are all contained between E and D. For by this you will have the Logarithms of all the Parts that are between t and E; and whose Logarithms lie between D and A: For Example, since O o is 9900 Parts, and its Logarithm 399563, the same Number may be taken for the Logarithm of 990, which is N n ; as also of the Number Y y 99, changing only the first Figure 3, because, according to the Composition of this Line, O N or N Y ought to be equal to E D or D C, as one may easily prove. So that O N, or N Y will contain 100000; and because A O is 399563, subtracting O N 100000, there will remain 299563, for A N; from whence also taking 100,000, there will rest 199563 for A Y. And after the same manner, having A Y 3995424 for the Logarithm of V u , which is 9000, you may have also 095424 for the Logarithm of X x , which is 9; or 195424 for the Logarithm of 90, or 29524 for the Logarithm of 900.

31. All this may be reduced to Practice for Calculation, without actually drawing these Figures, but only imagining them to be drawn. For by the Rules of Common Arithmetick we may find out F f , the mean Proportional between D d and E e , and after that, another Mean between D d and F f , or between F f and E e , &c. But what we have here explained is sufficient to gain as much Knowledge as is necessary for us to have of the Nature and Composition of Logarithms: There being no need for us to undergo the Labour of calculating Tables of Logarithms; since 'tis already so well and so often done to our Hands: God, for the Publick Good, having raised some Persons, whom he has pleased to endow with sufficient Patience to surmount so tedious and laborious

laborious a Work, as one would think to be insuperable. For we know that above 20 Men were engaged in such a Calculation, for above 20 Years together, with indefatigable Industry and Assiduity.

[*Pardie speaks here a little covertly, seeming willing to insinuate that this most useful and admirable Work was done first in his own Country; whereas the Logarithms were the Invention of my Lord Neper, a Scotch Baron, and the first Tables were calculated by him with the Assistance of our Countryman, Mr. Henry Briggs.*]

Of late several Improvements have been made in this Matter: As by Nicholas Mercator, of which see Dr. Wallis's Thoughts, in *Philosoph. Transact.* 38. John Gregory hath given us a Way to find Logarithms to 25 Places, by help of the Hyperbola. But Doctor Halley, in *Philos. Transf.* No. 216. shews a Way from the bare Consideration of Numbers, and without by the Help of Mr. Newton's Way to find the Unciæ of the Numbers of a Binomial Power, &c. By which you may find compendiously the Logarithms of all Numbers to above 30 Places. And he gives there several Series for this Purpose, some universal, and some appropriated to a peculiar sort of Logarithms.



ELEMENTS

OF

GEOMETRY.

BOOK IX.

Problems or Practical Geometry.



THAT Proposition is called a *Problem* in (*Geometry*) which teaches us how to do any Thing, and demonstrates also the Practice of it: Whereas *Theorems* are speculative Propositions, in which are considered the Affections and Properties of Things already done.

2. *To divide a Circle into four and into six, and all Arks into two equal Parts.* To divide it into four,



draw two Lines, as *dac* and *Bae* at Right Angles to each other. To divide it into eight Parts; bisect the four Arks *Bc*, *ce*, &c. which is done by striking (without the Ark *Bc*) two other Arks, with the same opening from the Points *B*, and *C*; for if a Line be drawn

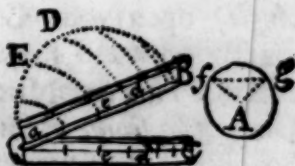
from the Point where those Arks cross each other, to the Center *a*, it shall bisect the Ark *BC*. The like is to be done for the other Arks.

To divide a Circle into six equal Parts; you need only take the Length of the Radius; and applying it six times about the Circle, it will exactly divide the Circumference into six equal Parts; and thus by a new Bisection, may a Circle be divided into 12, 24, 48, or into any Number of equal Parts, &c.

3. *To divide a Circle into five, into fifteen, and into other equal Parts.* This may be done thus; (as I demonstrate in *Algebra*) Make a Rectangled Triangle, one of whose Legs shall be the Radius of the Circle, and the other half the Radius. From the Hypothenuse of this Triangle, take half the Radius, the Remainder shall be the Chord of 36 deg. and the Side of a Decagon. Double that Ark, you have the Ark of 72 deg. (*whose Chord is the Side of a Pentagon*) and it is the fifth Part of the Circumference; and the same Chord shall be also the Hypothenuse of a Rectangled Triangle, one of whose Sides is the Radius, and the other the Side of a Decagon. And as by the last was found the Chord of 60 deg. so by subtracting the Chord of 36 deg. from 60 deg. you may have the Chord of 24 deg. which is the 15th Part of the Circumference. But, for Practice, the shortest and surest Way is, by repeated Trials with the Compasses to find a Distance

Distance that will go precisely five times about the Circle: Then divide, after the same manner (by Trials) that Distance into three equal Parts exactly. So shall you gain a Chord that will divide the Circumference into 15 equal Parts; and then dividing each of those 15 Chords into four equal Parts, and each of those into six, you will divide the whole Circumference into 360 *deg.* And this Division is most commodious for Practice and Use. Note, that the Way to divide a Circle into 3, 5, 7, or into any other odd Number of Parts, is not yet found Geometrically; Geometrically, I say, that is, by making Use only of a strait Line and Circle.

" This Division of a Circle into 360 *deg.* is very
 " useful, when a Person understands how to use the
 " *Compasses of Proportion* (or
 " *Sector*.) 'Tis so called, be-
 " cause 'tis a kind of Com-
 " passes with broad Legs:
 " As *a B*, *a C*, on which
 " are described diverse Lines
 " and Divisions, but those
 " which are most in Use, are of two Sorts. On one
 " Side of this Sector, and on each Leg, is a Line
 " *a e B* and *a e C*, which serves to divide a Cir-
 " cle into 360 *deg.* at once, and also to take at any
 " time as many Degrees as you please: And this
 " Line on the Sector is thus divided."

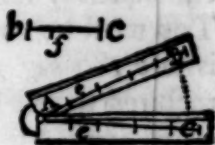


4. To divide and graduate the Sector, that it may serve for the Division of a Circle. Imagine a Semicircle *a E D B* accurately divided into 180 *deg.* if then from the Point *a*, as from a Center, you transfer the Divisions of the Semicircle into a Line *a B*; *v. gr.* If from *E*, 60 *deg.* you draw the Ark *E e*, and if from *D* 90 *deg.* in the Semicircle, you draw the Ark *D d*, &c. Then ought 60 *deg.* on that Leg of the Sector, to be placed at the Point *e*, and 90 *deg.*

deg. at the Point d , &c. And if you transfer the same Degrees after the same Manner into the other Leg aC , you will graduate the Lines aB and aC , (on the Sector) as they ought to be for this Purpose, *and they will be two similar Lines of Chords.*

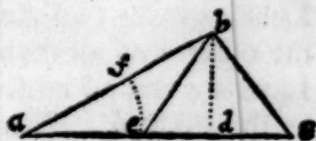
5. *To explain the Use of the Sector as far as it serves for the Division of a Circle.* Let there be a Circle given Af ; take with your Compasses the Radius Af , and (*keeping that Distance*) set one Foot of them in e or 60 *deg.* on one Leg of the Sector; move the other Leg of the Sector to and fro so long, till the other Point of the Compasses falls exactly on e or 60 *deg.* in that Leg of the Sector: So that the Distance ee be exactly equal to the Radius Af : Then if you would have readily 90 *degrees* of that Circle; (*letting the Sector lie still, and always keeping the same Angle*) open your Compasses 'till the Points fall exactly on d and d , or 90 *deg.* on each Side of the Sector: And then that Distance transferred into the Circle, in f, g , gives you the Ark of 90 *deg.* fg . So also if you would have had 35 *deg.* you need only apply your Compasses to 35 *deg.* and 35 *deg.* on each Leg of the Sector in the Lines (*of Chords*) aB and aC : and that Distance transferred into the Circle, shall cut off the Ark of 35 *deg.* and thus may you proceed to find any Degrees you please. All which is grounded on the 42, 43, 49 and 50 Propositions of the VIth Book. For since all Circles are similar Figures, (6. 50.) the Chord fg will be to the Radius fA : as the Chord of dd to the Radius ee ; that is, as ad is to de . Now 'tis plain, from what hath been proved elsewhere, that the Triangles add and nee are similar; and therefore $dd:ee::ad:ae$. But dd is by the Construction equal to fg , and ee to Af ; wherefore $fg:Af::da:ae$. Q. E. D.

6. To divide the Line of equal Parts or Lines on the Sector, for the Division of any Right-lines given. There being two Right-lines drawn from the Center of the Sector on the Legs, as aB and aC : Let each be divided into 100 or 200 equal Parts: And then they will serve to divide any Line given, into any Number of equal Parts: As for Instance, let the Line given be cb , and that you were required to take $\frac{25}{97}$ Parts of it. Now to divide the whole



Line cb into 97 equal Parts, and then to take 25 of them according to the common Way of dividing Lines, would be very tedious: But by the Sector 'tis done easily and speedily thus: Take the Length of the whole Line cb in your Compasses, and fit or apply it over in your Sector between 97 and 97 in each Leg, from B, suppose to C. Then letting the Sector lie open'd at that Angle, take in your Compasses, the Distance between 25 and 25 in each Leg, or between e and e , which transfer into the given Line from b to f ; so shall bf be just $\frac{25}{97}$ of the whole Line cb : As is plain from the Triangles ABC and Aee being similar.

7. On a Line given to make an Angle that shall contain any Number of Degrees assign'd. Let the Line given be ac , on which 'tis required to make an Angle of 30 deg. From the Point a , as from a Center, strike the Ark fe , from which take by the Sector, or otherwise, 30 deg. from e to f ; then through f draw the Line af , which with the Line ac will make an Angle of 30 deg.



8. Having

gle BCD : By which means you will have also the Angle BCA , because those two together make 180 deg. or two Right ones. So that in the Triangle ABC you have now found the Base AC , which is 10 Fathom, and also the two Angles at the Base; and consequently the Sides CB , and AB , may be known: (9. 18.) And then you may have the Height DB , or the Distance AD , if you make a little Triangle similar to that, and there from the Point b , let fall a Perpendicular bd , to the Base Line AC continued to d . For BD , or AD , will be just as many Fathoms as bd , and ad will be equal Parts measured on the Scale, (as in the last.) And if after you have thus gain'd the Height BD , you find by the same Method, the Height ED also, you may (by *Subtracting this Altitude from the former*) find the Height of the Tower EB .

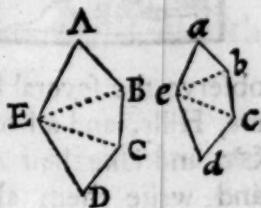
N. B. *The common Quadrant, with a String and Plummets, and with the Sights fix'd on one of its Sides, is more convenient and ready than this of Pardies's, which is now out of Use.*

“ Sometimes instead of advancing towards the
 “ Tower, and of making Observations of the Height
 “ below, or of those Angles the visual Rays make
 “ with the Horizontal Line, it is convenient to take
 “ two Stations side-ways of each other: But it
 “ comes all to the same, and the Methods in reality is not at all different.

“ And by this Means, as any one may see, may
 “ all imaginable Heights and Distances, and other
 “ Dimensions be taken; provided we can but come
 “ to observe their Extremities, from two different
 “ Places. I shall not stay now to describe the particular Way of doing this, nor to enumerate the
 “ great Advantages that would accrue from the Use

“ of Telescopical Sights fix'd on the Label, or on the
 “ Side of the Instrument us'd in taking Angles ;
 “ which indeed is an Invention of inestimable Bene-
 “ fit to Surveyors.

10. *To take the Plane of any Place.* Let ABCDE be a City, or any other Place, and you were required to take the Plane, and to make a Draught of it. Take all the Length of it Sides, and of Lines drawn from Angle to Angle: And transfer all these upon Paper laying them down according to their true Proportion. For Instance, having found AB to be 30 Paces, BC to be 59, CD to be 50, BE to be 67, and AE 49, &c. and having ready drawn on Paper, a plain Scale divided into 100 equal Parts: Make the Line *ab*, 30 of such Parts; *be*, 67; and *ae*, 49;

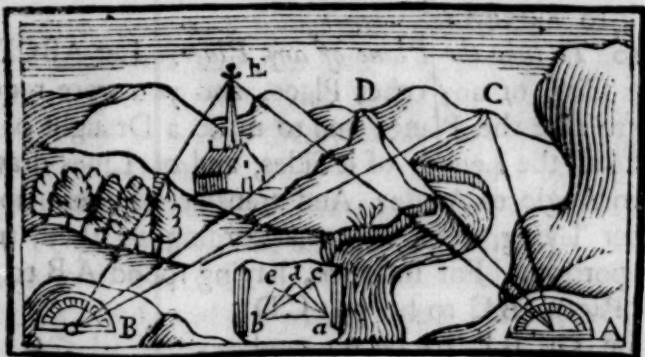


then those Lines drawn and join'd together will make the Triangle *abe* every way similar to the Triangle ABE. And if you go on thus, and make the Triangle *bec*, similar to BEC, &c. you will form the Figure *abcde* every way similar to the Plane of the Place ABCDE.

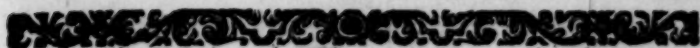
11. But if you cannot get into the Place to survey it, and to measure the Distance between the Angles FB and EC, you must take the several Angles of the Plane, and transfer them into your Draught; so that if the Angle BAE be 66 deg. the Angle *bae* must also be 66 deg. and so of all the rest.

12. *To make a Draught of any City or Country.* Ascend up into any two elevated Places, from whence you can plainly see the City and Country, whose Delineation you would make. And having with you a Quadrant, whole Circle, or Semicircle well divided into Degrees, together with it Labels (with Sights) and its Center: Place your Instrument at A, and

so that one of its Sides may lie in a Line between A and B, which done, and the Instrument fix'd there,



observe the several Steeples, eminent Houses, Towers, Hills, and all other remarkable Places, as EDC, &c. and take their Angles with the Label and Sights, and write them all down to help your Memory. Thus let the Angle CAB be 50 deg. 30 min. the Angle DAB 45 deg. 8 min. &c. Proceed after the same manner at the Station B; noting down the Angle ABC to be 40 deg. 10 min. the Angle ABD 47 deg. 28 min. &c. After which, draw on Paper any Line at Pleasure, as *ab*, and make, at each End of it, Angles equal to those which you found, *cab* equal to CAB, *dab* equal to DAB, and *abc* equal to ABD, &c. And by this Means you will have the Points *c*, *d* and *e*, &c. which will be in the same Position to one another as the Steeples, or other eminent Places CDE, &c. are. And thus having drawn the most conspicuous and principal Places, the rest may easily be taken by the Eye. But to make this Operation very exact, 'tis convenient to take the Angles also at a third or fourth Station, and then, if they all agree, any one will know that the Work was well done.



A

TABLE

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The Characters, Marks, Signs or Symbols here
used, are only these.

= Qual to.

+ E More, or Adding.

— Less, or Subtracting.

:: The Mark of four Quantities being discretely
proportional Geometrically.

∴ The Mark for Continual Proportion, or Geome-
trick Progression.

::: The Mark for Arithmetical Proportion.

× The Mark for Multiplication.

□ Square.

▭ Rectangle.

△ Triangle.

∠ Angle.

|| Parallel.

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